OPTIMAL MARTINGALE MEASURES AND HEDGING IN MODELS DRIVEN BY LÉVY PROCESSES

By

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I hereby declare that the work presented in this thesis was carried out by myself at Heriot-Watt University, Edinburgh, except where due acknowledgement is made, and has not been submitted for any other degree.

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Abstract

Our research falls into a broad area of pricing and hedging of contingent claims in incomplete markets. In the first part we introduce the Lévy processes as a suitable class of processes for financial modelling purposes. This in turn causes the market to become incomplete in general and therefore the martingale measure for the pricing/hedging purposes has to be chosen by introducing some subjective criteria.

We study several such criteria in the second section for a general stochastic volatility model driven by Lévy process, leading to minimal martingale measure, variance-optimal, or the more general q-optimal martingale measure, for which we show the convergence to the minimal entropy martingale measure for $q \downarrow 1$.

The martingale measures studied in the second section are put to use in the third section, where we consider various hedging problems in both martingale and semimartingale setting. We study locally risk-minimization hedging problem, mean-variance hedging and the more general p-optimal hedging, of which the mean-variance hedging is a special case for $p = 2$. Our model allows us to explicitly determine the variance-optimal martingale measure and the mean-variance hedging strategy using the structural results of Gourieroux, Laurent and Pham (1998) extended to discontinuous case by Arai (2005a).

Assuming a Markovian framework and appealing to the Feynman-Kac theorem, the optimal hedge can be found by solving a three-dimensional partial integro-differential equation. We illustrate this in the last section by considering the variance-optimal hedge of the European put option, and find the solution numerically by applying finite difference method.

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Chapter 1

Introduction

A Brownian motion is a stochastic process with independent, stationary increments of Gaussian distribution. It is without any doubt the most widely used process for the modelling of price fluctuations, whether considered in the Black-Scholes framework or more general diffusion models. The common property of these models is continuity, the property not observed in the price movement, as the prices move by jumps. However this is not the only reason for considering models with jumps. Many properties desirable from the models of price movement, either at the econometric or option pricing level, can only be obtained in diffusion models by considering very extreme parameters, while the same desirable properties can be easily obtained almost by definition when considering processes with jumps. Lévy processes represent one such class of processes. They share the common properties with Brownian motion that their increments are independent and stationary while having discontinuous paths in general. Lévy processes seem to offer the right balance between mathematical tractability and modelling possibilities. While many of the notions in this thesis can be considered in the more general setting of discontinuous semimartingales, considering a class of Lévy processes allows us to extend certain techniques not available for discontinuous processes in general or to get explicit results for the specific settings.

Introducing jumps into models creates a host of changes and new challenges. The indispensable Itô's formula changes, markets become incomplete in most of the cases and riskless hedging is no longer possible. A martingale measure, if exists, is no longer unique and therefore some additional subjective criteria have to be introduced to choose one that is used to perform hedging, leaving some risk that cannot be hedged away. Several criteria have been proposed, but so far there is none that would be preferred to others in all circumstances and in that way create an extension from pricing/hedging in the Black-Scholes framework in complete markets to incomplete ones.

Our research builds heavily on the paper of Chan (1999), where several martingale measures are studied in the framework of geometric Lévy processes. These include minimal, minimal entropy and Esscher transformed martingale measure. The variance-optimal martingale measure in this model is equal to the minimal martingale measure. We consider a model where this is no longer true. The model is based on a geometric Lévy process, that includes stochastic volatility driven by another independent Lévy process. This kind of model is interesting from both theoretical and practical aspects. From theoretical point of view, it allows us to find explicit form of the variance-optimal martingale measure, extending the work of Biagini, Guasoni and Pratelli (2000). This is then used in finding the mean-variance hedging strategy. The mean-variance hedging approach minimizes the expectation of the square difference between the value of the strategy and the underlying contingent claim at the maturity, among all self-financing strategies. This problem was mainly studied in two cases: when the price process is continuous, eg. Pham, Rheinländer and Schweizer (1998), Gourieroux, Laurent and Pham (1998), or under conditions that imply the equivalence of the minimal and the variance-optimal martingale measure, eg. Wiese (1998), Hubalek et al. (2006). When this thesis started, the mean-variance hedging problem in the general semimartingale setting was very active research area, which resulted in several new results. Arai (2005a), under some assumptions, extended results of Gourieroux et al. (1998) to a general semimartingale setting, Černý and Kallsen (2007) introduce opportunity process and opportunity martingale measure to tackle the problem, and other interesting works in this area include Lim (2005) or Xia. We used the result of Arai (2005a), but instead of assuming that the variance-optimal martingale measure is equivalent, we only assume that it is non-zero almost surely. The fact that we work with

Lévy processes allows us to get more explicit results. The optimal hedging strategy is given by the solution of the three dimensional partial integro-differential equation. We find the approximate solution to this problem by employing the finite difference method.

The thesis is structured as follows. In the first section we briefly review the main properties of Lévy processes that are used in the following chapters. We introduce the geometric Lévy process and the stochastic volatility model driven by independent Lévy processes. We then consider the absolutely continuous measure changes in these models, and obtain a set of (signed) martingale measures.

In the second chapter, we study various additional criteria that characterize martingale measure. We explicitly determine the variance-optimal martingale measure. Then we study more general q-optimal martingale measure ($q > 1$), and finish the chapter with the minimal entropy martingale measure.

We study the optimal hedging problem in the third chapter. The main body of this part concerns the mean-variance hedging problem. We consider various settings that illustrate complexity of this problem. The mean-variance hedging strategy is determined. We also consider q-optimal hedging problem when the price process is discontinuous but already a martingale.

The fourth chapter presents numerical results. In this section we specify both the price process and the variance process, and set up the finite difference method to obtain a numerical solution to the partial integro-differential equation, that is needed in order to find the mean variance hedging strategy. By simulating the processes, we then find approximate mean-variance hedge ratios corresponding to the given sample paths.

Chapter 2

Lévy processes in Finance

2.1 Introduction

In this chapter we briefly review the main properties of Lévy processes that are used in the following chapters. We introduce the geometric Lévy process and the stochastic volatility model driven by independent Lévy processes. We then consider the absolutely continuous measure changes in these models, and obtain a set of (signed) martingale measures. General reference to this section is the book by Cont and Tankov (2004), Applebaum (2004), or classic books by Jacod and Shiryaev (2002) and Sato (1999).

Definition 2.1. *(**Lévy process**) A stochastic process $X = (X_t)_{t\geq 0}$ with $X_0 = 0$ a.s. is called a Lévy process if it possesses the following properties:*

1. *Independent increments: $X_t - X_s$ is independent of $X_v - X_u$ if $(u, v) \cap (s, t) = \varnothing$.*

2. *Stationary increments: the law of $X_{t+h} - X_t$ does not depend on t.*

3. *Stochastic continuity: for all $\epsilon > 0, \lim_{h\to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \epsilon) = 0$.*

Every Lévy process can be characterized by a triplet (A, ν, γ), where A is a symmetric nonnegative definitive $d \times d$ matrix, $\gamma \in \mathbb{R}^d$ and ν is a Lévy measure, meaning that $\nu \in \mathbb{R}^d$ and satisfies

$$\nu(0) = 0 \qquad \text{and} \qquad \int_{\mathbb{R}^d} min(1, x^2)\nu(dx) < \infty.$$

The following theorem shows how the triplet (A, ν, γ) is used to calculate the characteristic function of a Lévy process.

Theorem 2.2. ***(The Lévy-Khintchine representation)*** *Let X be a Lévy process on $\mathbb{R}^d$ with characteristic triplet (A, ν, γ). Then*

$$\mathbb{E}\left[e^{i\langle z, X_t\rangle}\right] = e^{t\psi(z)}, z \in \mathbb{R}^d$$
$$\psi(z) = -\frac{1}{2}\langle z, Az\rangle + i\langle\gamma, z\rangle + \int_{\mathbb{R}^d}(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_{|x|\leq 1})\nu(dx). \quad (2.1)$$

Proof. cf. Cont and Tankov (2004), proof of Theorem 3.1. □

Let J be a Poisson random measure associated to the jump process $\triangle X$ of the Lévy process X. Its compensated measure is defined by

$$N(dt, dx) = J(dt, dx) - \nu(dx)dt. \quad (2.2)$$

The next theorem shows that the sample paths of Lévy processes can be decomposed into continuous and jump parts.

Theorem 2.3. ***(The Lévy-Itô decomposition)*** *Let X be a Lévy process. There exists $b \in \mathbb{R}^d$, called the drift of the Lévy process, a Brownian motion B_A with covariance matrix A and an independent Poisson random measure J on $\mathbb{R}_+ \times (\mathbb{R}^d - 0)$ such that, for every $t > 0$,*

$$X(t) = bt + B_A(t) + \int_{|x|<1} xN((0, t], dx) + \int_{|x|\geq 1} xJ((0, t], dx). \quad (2.3)$$

Proof. cf. Applebaum (2004), proof of Theorem 2.4.16. □

Let us note that this decomposition is unique and

$$b = \mathbb{E}\left(X(1) - \int_{|x|\geq 1} xJ((0, t], dx)\right).$$

Definition 2.4. ***(The Lévy stochastic integral)*** *Let X be a Lévy process with characteristics (A, ν, γ) and Lévy-Itô decomposition given by (2.3). Let $L = (L(t), T \geq 0)$ be a square integrable predictable process, then we can construct processes with the stochastic differential*

$$dY(t) = L(t)dX(t). \quad (2.4)$$

The process Y is called a Lévy stochastic integral.

Most of the stochastic integrals we consider will be one-dimensional Lévy-type stochastic integrals of the following form

$$dY_t = H_t dB_t + \int_{\mathbb{R}} h(t,x)N(dt,dx), \tag{2.5}$$

where B is now a one-dimensional standard Brownian motion. We will denote by Y^c the continuous part of Y.

Theorem 2.5. ***(Itô's formula)*** *Let Y be a Lévy-type stochastic integral of the form* (2.5)*. For any $f \in C^2(\mathbb{R})$, $t > 0$, with probability* 1 *we have*

$$\begin{aligned}
f(Y_t) - f(Y_0) &= \int_0^t \partial f(Y_{u-}) dY_u^c + \frac{1}{2}\int_0^t \partial^2 f(Y_{u-}) d[Y^c, Y^c]_u \\
&\quad + \int_0^t \int_{\mathbb{R}} \big(f(Y_{u-} + h(u,x)) - f(Y_{u-})\big) N(du,dx) \\
&\quad + \int_0^t \int_{\mathbb{R}} \big(f(Y_{u-} + h(u,x)) - f(Y_{u-}) - h(u,x)\partial f(Y_{u-})\big)\nu(dx)du \\
&= \int_0^t \partial f(Y_{u-}) dY_u + \frac{1}{2}\int_0^t \partial^2 f(Y_{u-}) d[Y^c, Y^c]_u \\
&\quad + \sum_{0 \le u \le t} \big(f(Y_u) - f(Y_{u-}) - \triangle Y_u \partial f(Y_{u-})\big).
\end{aligned} \tag{2.6}$$

Proof. cf. Applebaum (2004), proof of Theorems 4.4.7 and 4.4.10. □

Remark 2.1. The second form of Itô's formula is more general and holds for any semimartingale. When the Lévy process has finite number of jumps, i.e. $\nu(\mathbb{R}) < \infty$, one can rewrite equation (2.6) to the following form

$$\begin{aligned}
f(Y_t) - f(Y_0) &= \int_0^t \partial f(Y_{u-}) dY_u^c + \frac{1}{2}\int_0^t \partial^2 f(Y_{u-}) d[Y^c, Y^c]_u \\
&\quad + \sum_{0 \le u \le t} \big(f(Y_u) - f(Y_{u-})\big).
\end{aligned}$$

Theorem 2.6. ***(Martingale representation for Lévy processes)*** *Let M be a local martingale adapted to the filtration generated by the Lévy process X. Then there exist unique pair of square integrable processes (ϕ_s, ψ_s) such, that M_t is represented as*

$$M_t = M_0 + \int_0^t \phi_s dB_s + \int_0^t \int_{\mathbb{R}} \psi(s,x) N(ds,dx), \tag{2.7}$$

where B is a standard Brownian motion and N is a compensated Poisson random measure of the Lévy process X.

Proof. cf. Kunita (2004), proof of Theorem 1.1. □

Definition 2.7. ***(Orthogonality in the martingale sense)*** *Two locally square integrable martingales M^1 and M^2 are called (strongly) orthogonal if $M^1 M^2$ is a local martingale, denoted by $M^1 \perp\!\!\!\perp M^2$.*

Note that there is also a notion of weak orthogonality of martingales, but when we will talk about orthogonal processes, we mean orthogonality in the strong sense. We will find the following results useful.

Proposition 2.8. ***(Independence of Lévy processes)*** *Let (X_t, Y_t) be a Lévy process with Lévy measure ν and without Gaussian part. Its components are independent if and only if the support of ν is contained in the set $\{(x, y) : xy = 0\}$, that is, if and only if they never jump together. In this case*

$$\nu(A) = \nu_X(A_X) + \nu_Y(A_Y) \tag{2.8}$$

where $A_X = \{x : (x, 0) \in A\}$ and $A_Y = \{y : (0, y) \in A\}$, and ν_X and ν_Y are Lévy measures of (X_t) and (Y_t).

Proof. cf. Cont and Tankov (2004), proof of Proposition 5.3. □

Definition 2.9. ***(Quadratic covariation)*** *Let X and Y be two semimartingales. The quadratic covariation process $[X, Y]$ is the semimartingale defined by*

$$[X, Y]_t = X_t Y_t - X_0 Y_0 - \int_0^t X_{u-} dY_u - \int_0^t Y_{u-} dX_u. \tag{2.9}$$

Definition 2.10. ***(Conditional quadratic covariation)*** *Let X and Y be two semimartingales, and $[X, Y]$ is locally of integrable variation. Then the conditional quadratic covariation $\langle X, Y\rangle$ exists and is defined to be the compensator of $[X, Y]$.*

The process $[X, Y]$ is also called *square bracket*, and the process $\langle X, Y\rangle$ *sharp* or *angle bracket*.

Example 2.1. *Let $M^1(t)$ and $M^2(t)$ be two local martingales with representation kernels (ϕ^1, ψ^1) and (ϕ^2, ψ^2) from Theorem 2.6 respectively. Then*

$$[M^1, M^2]_t = \int_0^t \phi_s^1 \phi_s^2 ds + \int_0^t \int_{\mathbb{R}} \psi^1(s, x)\psi^2(s, x) J(ds, dx), \tag{2.10}$$

and

$$\langle M^1, M^2\rangle_t = \int_0^t \phi_s^1 \phi_s^2 ds + \int_0^t \int_{\mathbb{R}} \psi^1(s, x)\psi^2(s, x) \nu(ds, dx). \tag{2.11}$$

2.2 Stochastic exponential and exponential martingale

Let X be a general semimartingale. The adapted process that is a solution of the following stochastic integral equation

$$Z = 1 + \int Z_{-} dX, \qquad Z_0 = 1, \tag{2.12}$$

is called *stochastic exponential* or *Doléans-Dade exponential*. The continuous part of the semimartingale X is denoted by X^c. The solution to the equation (2.12) is given by

$$Z_t = \exp\left(X(t) - \frac{1}{2}[X, X]^c(t)\right) \prod_{0 \le s \le t} [1 + \triangle X(s)] e^{-\triangle X(s)} \tag{2.13}$$

for each $t \geq 0$, and often denoted by $\mathcal{E}(X)$. When we define equivalent probability measure changes, we will use the following condition

$$\inf\{\triangle X(t), t > 0\} > -1 \qquad \text{a.s.},$$

which guarantees positivity of the stochastic exponential. However, many interesting "optimal" measures are only signed in general when dealing with processes with jumps, and for those we instead assume

$$\triangle X(t) \neq -1 \qquad \text{a.s. for } 0 \leq t \leq T.$$

In that case we can use the following characterization of semimartingales by stochastic exponentials.

Proposition 2.11. *Let Z be a semimartingale. There exists a semimartingale X such that $Z = \mathcal{E}(X)$ if and only if $Z_0 = 1$, $Z_t \neq 0$ a.s. for $0 \leq t \leq T$. In that case we can choose $X := Z_{-}^{-1} Z$. The process X is called stochastic logarithm of Z.*

Proof. cf. Jacod (1979), proof of Proposition 6.5 and Exercise 6.1. □

We can see from equation (2.12), that the stochastic exponential of a martingale is a local martingale. For a martingale Lévy process, the stochastic exponential is even a true martingale, cf. Proposition 1.4 in Tankov (2004). This will be important because for Z_T to be a density of a martingale measure, it needs to be a true martingale, not just a local martingale. If Z were only a local martingale measure, it could lead to pricing irregularities when pricing via expectation technique is used, cf. Cox and Hobson (2005). It is also equally important in the change of numéraire technique, where the numéraire is required to be a true martingale as well. The following Lemma will be useful as it provides an extension of Novikov condition for discontinuous processes, cf. Rheinländer and Steiger (2006), Lemma 2.11.

Lemma 2.12. *Let M be a locally bounded local $\mathbb{P}$-martingale, and let $Z_t = \mathcal{E}(M)_t$ where $\triangle M > -1$. If the process*

$$\frac{1}{2}\langle M^c\rangle_t + \sum_{s\leq t}\big\{(1+\triangle M_s)\log(1+\triangle M_s) - \triangle M_s\big\}$$

has a predictable compensator L_t satisfying $\mathbb{E}[\exp(L_T)] < \infty$, then Z is a true martingale.

However, note that the above Lemma is only useful when considering the class of probability measures as $\triangle M > -1$ which assures that $Z > 0$.

Another interesting problem is to find a Lévy process X such that e^X is a martingale, then called an *exponential martingale*. As it will be seen in the next section, there are various ways to define models with jumps and this will show the equivalence between some of these models and a way to switch from one to another. The following Lemma, due to Goll and Kallsen (2000), states a sufficient condition so that the stochastic exponential of one Lévy process equals the exponential martingale of another Lévy process. Let Λ be any Borel set $\Lambda \subset \mathbb{R}\backslash\{0\}$.

Lemma 2.13. *1. Let $\{X_t\}_{t\geq 0}$ be a real-valued Lévy process with characteristic triplet (A,ν,γ) and $Z = \mathcal{E}(X)$ its stochastic exponential. If $Z > 0$ a.s. then there exists another Lévy process $\{L_t\}_{t\geq 0}$ such that for all t, $Z_t = e^{L_t}$. The process L is given by*

$$L_t = \log Z_t = X_t - \frac{At}{2} + \sum_{0\leq s\leq t}\{\log(1+\triangle X_s) - \triangle X_s\}, \qquad t\geq 0.$$

Its stochastic triplet (A_L, ν_L, γ_L) is given by

$$
\begin{aligned}
A_L &= A, \\
\nu_L(\Lambda) &= \nu(\{x : \log(1+x)\} \in \Lambda) = \int_{-\infty}^{\infty} 1_\Lambda(\log(1+x))\nu(dx) \\
\gamma_L &= \gamma - \frac{A}{2} + \int_{-\infty}^{\infty} \nu(dx)\{\log(1+x)1_{[-1,1]}(\log(1+x)) - x1_{[-1,1]}(x)\}.
\end{aligned}
$$

2. *Let $\{L_t\}_{t\geq 0}$ be a real-valued Lévy process with characteristic triplet (A_L, ν_L, γ_L) and $S_t = e^{L_t}$ its exponential. Then there exists another Lévy process $\{X_t\}_{t\geq 0}$ such that $S = \mathcal{E}(X)$, where*

$$X_t = L_t - \frac{At}{2} + \sum_{0\leq s\leq t} \{e^{\triangle X_s} - 1 - \triangle L_s\}$$

Its stochastic triplet (A_L, ν_L, γ_L) is given by:

$$
\begin{aligned}
A &= A_L, \\
\nu(\Lambda) &= \nu_L(\{x : e^x - 1\} \in \Lambda) = \int_{-\infty}^{\infty} 1_\Lambda(e^x - 1)\nu_L(dx), \\
\gamma &= \gamma_L + \frac{A_L}{2} + \int_{-\infty}^{\infty} \nu_L(dx)\{(e^x - 1)1_{[-1,1]}(e^x - 1) - x1_{[-1,1]}(x)\}.
\end{aligned}
$$

Proof. cf. Goll and Kallsen (2000), proof of Lemma A.8. □

2.3 Models

Our research started off by studying the paper of Chan (1999), which provides excellent introduction for measure changes in the context of Lévy processes. In this paper both the minimal and the minimal entropy martingale measure are determined, and provide building blocks to study the variance-optimal and the more general q-optimal martingale measure. The price process S in Chan (1999) is modelled by a geometric Lévy process

$$dS_t = S_{t-}\{\sigma_t dX_t + b_t dt\}, \tag{2.14}$$

where σ_t and b_t are deterministic continuous functions of time, and X is a general Lévy process. The interest rates are assumed deterministic and S represents the discounted price of stocks. Using the Lévy-Itô decomposition, cf. Theorem 2.3, X

can by separated into its Brownian part denoted by B and the quadratic pure jump part J by writing $X_t = cB_t + J_t$, for some $c \in \mathbb{R}$. We make the following standing

Assumption 2.1. *The Lévy process X satisfies $\int_{|x|\geq 1} x\nu(dx) < \infty$.*

The above assumption, together with the main assumption on the Lévy density, can be written as $\int(|x|^2 \wedge |x|)\nu(dx) < \infty$. This will rule out Lévy process without first moment, for example α-stable Lévy process with $\alpha < 1$. But it means that there is no need to truncate large jumps. This will simplify a lot of the formulas, because it implies that one can take x as the truncation function, as opposed to the usual truncation function $1_{|x|\leq 1}$. In fact, X is then a special semimartingale.

Lemma 2.14. *A Lévy process X is a special semimartingale if and only if it is integrable, i.e. $\mathbb{E}[X_1] < \infty$.*

Proof. cf. Kallsen (1998), proof of Lemma 2.2, 2. □

Under the Assumption 2.1, the Doob-Meyer decomposition of J is given by

$$J_t = N_t + at, \tag{2.15}$$

and using equation (2.2), we have

$$N_t = \int_{\mathbb{R}} x\big(J((0,t], dx) - t\nu(dx)\big), \qquad a = \mathbb{E}(J_1).$$

The Lévy-Khintchine formula (2.1) becomes

$$\psi(z) = -\frac{c^2 z^2}{2} + iaz + \int_{\mathbb{R}} (e^{izx} - 1 - izx)\nu(dx).$$

Thus, we can rewrite (2.14) into

$$dS_t = S_{t-}\{\sigma_t(cdB_t + dN_t) + (a\sigma_t + b_t)dt\}. \tag{2.16}$$

An explicit solution to equation (2.16) is given by

$$S_t = S_0 \exp\left\{\int_0^t c\sigma_s dB_s + \int_0^t \sigma_s dN_s + \int_0^t \left(a\sigma_s + b_s - \frac{c^2\sigma_s^2}{2}\right) ds\right\}$$
$$\times \prod_{0<s\leq t} (1 + \sigma_s \triangle N_s) \exp(-\sigma_s \triangle N_s),$$

or using stochastic exponential form

$$S_t = S_0 \mathcal{E} \left(\int_0^t (a\sigma_s + b_s) ds + \int_0^t c\sigma_s dB_s + \int_0^t \sigma_s dN_s \right).$$

We now extend model (2.16) to include stochastic volatility, possibly with jumps as well:

$$\begin{aligned} dS_t &= S_{t-}\big\{ b(t, V_{t-})dt + \sigma(t, V_{t-})dB_t + \delta(t, V_{t-})dJ_t \big\} \\ dV_t &= g(t, V_{t-})dt + \gamma(t, V_{t-})dL_t. \end{aligned} \tag{2.17}$$

Here J is a pure jump Lévy process, and L is another Lévy process independent of J and of the Wiener process B. We again assume that Lévy processes J and L satisfy Assumption 2.1. For the price process S to remain non-negative, we need $\delta(t, V_{t-})\triangle J_t \geq -1$ for all t. Let the Lévy measure of J be supported on $[-c_1, c_2]$ for $c_1, c_2 > 0$. Thus, in order for S to remain non-negative

$$-\frac{1}{c_2} \leq \delta(t, V_{t-}) \leq \frac{1}{c_1}, \qquad \text{for all } t,$$

which implies that the jumps of J need to be bounded from below, i.e. $c_1 < \infty$, when $\delta(t, V_{t-}) > 0$ for some t and bounded from above, i.e. $c_2 < \infty$, when $\delta(t, V_{t-}) < 0$ for some t. If δ changes sign in $[0, T]$, then J needs to have compact support.

Unless stated otherwise, we assume that $\delta(t, \cdot) \neq 0$ and $\sigma(t, \cdot) \neq 0$ for all $t \in [0, T]$ and $\nu(\mathbb{R}\backslash 0) \neq 0$. We also implicitly assume that model parameters have sufficient regularity properties so that there exists unique solution to (2.17) which does not explode on $[0, T]$, see Theorem V.38 in Protter (2004), and Assumption 3.1 in Rheinländer and Steiger (2006) for general jump-diffusion model with stochastic volatility.

From now on, the quantities related to the volatility will be denoted by superscript V. For example, the Lévy-Itô decomposition of L is given by

$$\begin{aligned} L_t &= c^V B_t^V + J_t^V \\ &= c^V B_t^V + \int_0^t \int_{\mathbb{R}} x N^V(dt, dx) + a^V t. \end{aligned}$$

The stock S is the only traded asset in this market. Such a market is incomplete and there are infinitely many equivalent martingale measures. Note, that market is already incomplete in the case of model (2.16) even without using stochastic volatility. The reason for considering this kind of model is manifold. The model encompasses many uncorrelated stochastic volatility models, for example uncorrelated Heston model, Stein and Stein model or Hull-White model are all special cases of the model considered here. From the practical point of view, the question one might ask is why to consider model that combines Lévy process with stochastic volatility. Does model with Lévy processes not provide enough flexibility over the standard geometric Brownian motion? It is well known that stochastic volatility models work best for mid to long term options. They have problems at short maturities as the Gaussian-based stochastic volatility models in general produce shallow implied volatility smiles, which does not correspond with the observed market implied smiles, see Chapter 8 in Rebonato (2004) and references therein. One reason for this is that convexity of the smile depends to some extent on the speed at which the volatility moves from its current value, which is in general not sufficient for reasonable values of volatility of volatility. On the other hand, models with jumps work better for short term options, while failing at long terms. From the calibration point of view, the combination of a Lévy driven model and stochastic volatility allows fitting to the implied volatility surface without needing time-dependent model parameters, see Cont and Tankov (2004). Also, Li, Wells and Yu (2006) in their study of S&P 500 returns provide an economic justification for the class of stochastic volatility models driven by infinite activity Lévy processes. They show superiority of such models in capturing the index returns over even the most sophisticated affine jump diffusion models. From theoretical point of view, the reason for considering this kind of model will become more apparent when we introduce mean-variance tradeoff process and later mean-variance hedging.

2.4 Absolutely continuous measure changes

We follow Chan (1999) and provide a characterization of all the measures $\mathbb{Q}$ that are absolutely continuous with respect to the physical measure $\mathbb{P}$. We do this first for a geometric Lévy model (2.16) and then extrapolate to the stochastic volatility model driven by Lévy processes (2.17).

Define a process Z by

$$\begin{aligned} Z_t &= \exp\left\{\int_0^t H_s dB_s - \frac{1}{2}\int_0^t H_s^2 ds + \int_0^t\int_{\mathbb{R}} h(s,x)N(ds,dx)\right\} \\ &\quad\times \prod_{0<s\le t}\big(h(s,\triangle J_s)+1\big)\exp\big(-h(s,\triangle J_s)\big) \\ &= \mathcal{E}\left(\int H_s dB_s + \iint_{\mathbb{R}} h(s,x)N(ds,dx)\right)_t \qquad t\ge 0, \end{aligned} \tag{2.18}$$

where H is a previsible square integrable process and $h(t,x)$ is a Borel previsible process satisfying $h(t,0)=0$ for all $t\ge 0$.

Lemma 2.15. *The process Z defined by* (2.18) *is a local martingale with $Z_0 = 1$ and Z is positive if and only if $h > -1$.*

Proof. cf. Chan (1999), proof of Lemma 3.1. □

Theorem 2.16. *Let $\mathbb{Q}$ be an absolutely continuous measure with respect to $\mathbb{P}$ on $\mathcal{F}_T$. Then there exists a martingale Z satisfying equation* (2.18) *such that*

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = Z_T,$$

The variables H and h are chosen such that $\mathbb{E}[Z_T]=1$.

Proof. cf. Chan (1999), proof of Theorem 3.2. □

Remark 2.2. If $h>-1$, Z is a positive local martingale and thus a supermartingale (a consequence of Fatou's lemma). Moreover, if the process h in Lemma 2.15 is defined such that $\mathbb{E}[Z_t]=1$ for all t, then Z is a true martingale.

Let $(\mathcal{F}_t)$, $t\in[0,T]$ be the filtration generated by the Brownian motion B_t and the Poisson random measure $J(dt,dx)$. With respect to $(\mathcal{F}_t,\mathbb{Q})$ we have

1) The process

$$\tilde{B}_t = B_t - \int_0^t H_s ds \tag{2.19}$$

is a standard Brownian motion.

2) The process J is a quadratic pure jump process with the compensator measure

$$\tilde{\nu}(dt,dx) = \big(h(t,x)+1\big)dt\nu(dx), \tag{2.20}$$

that is

$$\tilde{N}(dt,dx) := J(dt,dx) - \tilde{\nu}(dt,dx) \tag{2.21}$$

is a martingale, or equivalently writing

$$\tilde{N}_t = N_t - \int_0^t \int_{\mathbb{R}} x h(s,x)\nu(dx)ds. \tag{2.22}$$

3) Let $M(t)$ be a $\mathbb{Q}$-local martingale. Then there exists a pair of predictable processes $(\phi_t, \psi(t,x))$ such that M_t is represented by

$$M_t = M_0 + \int_0^t \phi_s d\tilde{B}_s + \int_0^t \int_{\mathbb{R}} \psi(s,x)\tilde{N}(ds,dx). \tag{2.23}$$

Remark 2.3. $J(dt,dx)$ is no longer a Poisson random measure with respect to $\mathbb{Q}$ unless h is a deterministic function. Moreover, if h is deterministic but time-dependent, then increments of J from equation (2.15) under $\mathbb{Q}$ will be independent, but not stationary. If h is stochastic, the increments of J will be dependent under measure $\mathbb{Q}$.

Remark 2.4. In other words, Z_T defines the *Radon-Nikodým derivative* of $\mathbb{Q}$ with respect to $\mathbb{P}$. Also, note that T has to be less than infinity, which is implied by the measure $\mathbb{Q}$ being absolutely continuous with respect to $\mathbb{P}$, otherwise the *Radon-Nikodým derivative* would be either zero or infinity, implying that the measures $\mathbb{P}$ and $\mathbb{Q}$ are mutually singular.

The following theorem will be used whenever we move from one measure to another. It provides a sufficient condition that allows separation of the jump measure from its compensator.

Theorem 2.17. *If the increasing process* $\iint_{\mathbb{R}} |\psi(s,x)|J(ds,dx)$ *(or* $\iint_{\mathbb{R}} |\psi(s,x)|\nu(ds,dx)$*) is locally* $\mathbb{P}$*-integrable, then* ψ *is integrable with respect to the compensated measure and*

$$\iint_{\mathbb{R}} \psi(s,x)N(ds,dx) = \iint_{\mathbb{R}} \psi(s,x)J(ds,dx) - \iint_{\mathbb{R}} \psi(s,x)\nu(ds,dx).$$

Proof. cf. Jacod and Shiryaev (2002), Proposition II.1.28. □

Using now any equivalent martingale measure $\mathbb{Q}$ given by Theorem 2.16, equation (2.19) for the Brownian motion $\tilde{B}$ and equation (2.22) for the compensated measure

$\tilde{N}$, under $\mathbb{Q}$, the price process S is given by

$$S_t = S_0 \exp\left\{\int_0^t c\sigma_s d\tilde{B}_s + \int_0^t \sigma_s d\tilde{N}_s + \int_0^t \left(a\sigma_s + c\sigma_s H_s + b_s - \frac{c^2\sigma_s^2}{2}\right) ds \right.$$
$$\left. + \int_0^t \sigma_s \int_{\mathbb{R}} xh(s,x)\nu(dx)ds\right\} \prod_{0<s\le t} (1+\sigma_s \triangle\tilde{N}_s)\exp(-\sigma_s\triangle\tilde{N}_s).$$

Looking at the form of the equation above, we deduce that the general condition for the process S to be a martingale under $\mathbb{Q}$ is

$$c\sigma_s H_s + a\sigma_s + b_s + \int_{\mathbb{R}} \sigma_s x h(s,x)\nu(dx) = 0 \qquad \text{for all } s \text{ a.s.} \tag{2.24}$$

It is clear that H and h are not given uniquely by the martingale condition (2.24). This is one way of seeing that the geometric Lévy model is incomplete. We will study various approaches in the next section that, in addition to (2.24), allow to specify H and h.

Moving onto the stochastic volatility model (2.17), similarly to (2.18), define the process Z by

$$Z_t = \mathcal{E}\left(\int H_u dB_u + \iint_{\mathbb{R}} h(u,x)N(du,dx) + \right.$$
$$\left. \int F_u dB_u^V + \iint_{\mathbb{R}} f(u,x)N^V(ds,dx)\right)_t, \tag{2.25}$$

where H, F are previsible square integrable processes and $h(t,x)$, $f(t,x)$ are previsible processes satisfying $h(t,0)=0$ and $f(t,0)=0$ for all $t \geq 0$ and further assume that $\{h(t,x) = -1\}$ and $\{h(t,x) = -1\}$ are evanescent. The martingale condition practically looks the same as for the geometric Lévy process, but there is now much more freedom due to F and f from (2.25) not being present in the following martingale condition:

$$\sigma(s,V_{s-})H_s + b(s,V_{s-}) + \delta(s,V_{s-})\left(a + \int_{\mathbb{R}} xh(s,x)\nu(dx)\right) = 0 \qquad \text{for all } s \text{ a.s.} \tag{2.26}$$

There is an alternative way to express the density of a martingale measure. Assuming that S is a special semimartingale (sufficient condition for this is for example $\mathbb{E}(|S_t|) < \infty$), S has a unique decomposition

$$S = M + A,$$

where M is a local martingale and A is a predictable process of finite variation. See Example 3.1 for the form of M and A in stochastic volatility model given by (2.17). It then follows, by the absence of arbitrage, that the finite variation part A must be absolutely continuous with respect to the predictable quadratic variation of the martingale part M. This implies that there exists a predictable process $\hat{\lambda}$ such that

$$A_t = \int_0^t \hat{\lambda}_s d\langle M \rangle_s,$$

and this is usually termed as the *structure condition.* The following is a one dimensional version of the second claim in Proposition 2 of Schweizer (1995).

Proposition 2.18. *Suppose that S satisfies the structure condition. A square-integrable local martingale Z is a martingale density for S if and only if Z satisfies the stochastic differential equation*

$$Z_t = 1 - \int_0^t \hat{\lambda}_u Z_{u-} dM_u + R_t, \qquad 0 \leq t \leq T \tag{2.27}$$

for some square-integrable local martingale R strongly orthogonal to M.

Proof. cf. Yoeurp and Yor (1977), proof of Theorem 2.1. □

Remark 2.5. When both S and R are assumed to be locally bounded, the representation (2.27) is sufficient to ensure that $\mathbb{Q}$ is a true martingale, see Corollary 3.2.2 of Steiger (2005). However, in the present context of Lévy processes, S and R are in general not locally bounded.

By the martingale representation property, R can be written as

$$R_t = \int_0^t \sigma_u^R dB_u + \int_0^t \int_{\mathbb{R}} \delta^R(u,x) N(du,dx),$$

and as R is orthogonal to M, it satisfies

$$\langle M, R \rangle_t = \int_0^t \sigma_u \sigma_u^R du + \int_0^t \int_{\mathbb{R}} \delta_u \delta^R(u,x) x N(du,dx) = 0. \tag{2.28}$$

Thus the alternative way to express the density of the martingale measure by equation (2.25) satisfying the martingale condition (2.26) is given by equation (2.27) satisfying the orthogonality condition (2.28).

Chapter 3

Optimal martingale measures

3.1 Introduction

In the previous chapter, we have seen that the martingale condition (2.24) or (2.26) is not enough to identify a martingale measure uniquely. In this chapter, we study various additional criteria that characterize martingale measure in addition to the martingale condition. First, we determine the Föllmer-Schweizer minimal martingale measure. We then explicitly determine the variance-optimal martingale measure and discuss cases when it is equal to the minimal martingale measure. This will have consequences in the next chapter where we study mean-variance hedging problem. We finish the chapter with the q-optimal martingale measure ($q > 1$), which is a generalization of the variance-optimal martingale measure for which $q = 2$.

We denote by Θ some space of S-integrable predictable processes (further properties and requirements on Θ will be discussed later). For a positive time horizon T, the stochastic integral $G_T(\vartheta) = \int_0^T \vartheta_t dS_t$ represents gains from trading according to ϑ, which itself can be regarded as a self-financing trading strategy.

Definition 3.1. *A signed martingale measure is a signed measure $Q \ll P$ with $\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = 1$ and $\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} G_T(\vartheta)\right] = 0$ for all $\vartheta \in \Theta$.*

The space of all signed martingale measures will be denoted by $\mathscr{M}^s(\mathbb{P})$. The subset of $\mathscr{M}^s(\mathbb{P})$ of probability measures that are absolutely continuous martingale measures will be denoted by $\mathscr{M}(\mathbb{P})$, and its subset of equivalent martingale measures will be denoted by $\mathscr{M}^e(\mathbb{P})$:

$$\mathscr{M}(\mathbb{P}) = \{\mathbb{Q} \ll \mathbb{P} : \mathbb{Q} \text{ is a probability measure and } S \text{ is a } \mathbb{Q}\text{-local martingale}\},$$
$$\mathscr{M}^e(\mathbb{P}) = \{\mathbb{Q} \sim \mathbb{P} : \mathbb{Q} \text{ is a probability measure and } S \text{ is a } \mathbb{Q}\text{-local martingale}\}.$$

While not explicitly used in notation, note that $\mathscr{M}^s$, $\mathscr{M}$ and $\mathscr{M}^e$ depend on the choice of space Θ, and we need to assume that Θ contains all indicator processes $1_{[t_1,t_2[}$.

3.2 The minimal martingale measure

Recall that S denotes the discounted price of a stock and we assume that S is a special semimartingale with a Doob-Meyer decomposition of the form $S = S_0 + M + A$, where M is a local martingale and A is a process of finite variation. Under this assumption, there exists a process $\hat{\lambda}$ such that $A = \int \hat{\lambda} d\langle M\rangle$ and the following process

$$\hat{K} = \int \hat{\lambda} dA = \int \hat{\lambda}^2 d\langle M\rangle$$

is called the mean-variance tradeoff process. This is nothing else than the integrated squared market price of risk. Using these notions, we can define the minimal martingale measure introduced in Föllmer and Schweizer (1991). The density of the minimal martingale measure is given by

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}\left(-\int \hat{\lambda} dM\right)_T$$

providing it exists. This is the case when $\mathcal{E}\left(-\int \hat{\lambda} dM\right)$ is a uniformly integrable martingale. The "minimal" in the name refers to the fact, that the change of measure preserves the structure of the reference measure $\mathbb{P}$ as much as possible in the following sense: if a square-integrable $\mathbb{P}$-martingale is orthogonal to M, then it is a $\hat{\mathbb{P}}$-martingale as well. For continuous processes, the minimal martingale measure also preserves orthogonality: if a square-integrable $\mathbb{P}$-martingale is orthogonal to M, then it is orthogonal to S under $\hat{\mathbb{P}}$ as well. However, this is no longer true when M is not continuous, and this will play some role when we later discuss the mean-variance and local risk-minimization hedging problems. Note also, that when the process S has jumps, $\mathcal{E}\left(-\int \hat{\lambda} dM\right)$ may become negative, and this means that the minimal martingale measure is only a signed measure in general. For further properties of the minimal martingale measure, see Schweizer (1995).

The minimal martingale measure for the geometric Lévy process was determined in Chan (1999). Here, we determine the minimal martingale measure for the stochastic volatility model (2.17).

Example 3.1. *Stochastic volatility model. The Doob-Meyer decomposition of S is given by*

$$\begin{aligned} M_t &= \int_0^t S_{u-}\left(\sigma(u, V_{u-})dB_u + \delta(u, V_{u-})\int_{\mathbb{R}} xN(dt, dx)\right), \\ A_t &= \int_0^t S_{u-}\big(\delta(u, V_{u-})a + b(u, V_{u-})\big)du. \end{aligned}$$

The mean-variance tradeoff process $\hat{K}$ is given by

$$\hat{K}_t = \int_0^t \hat{\lambda}_u dA_u = \int_0^t \frac{\big(\delta(u, V_{u-})a + b(u, V_{u-})\big)^2}{\sigma(u, V_{u-})^2 + \delta(u, V_{u-})^2 \int_{\mathbb{R}} x^2 \nu(dx)} du, \tag{3.1}$$

where

$$\hat{\lambda}_t = \frac{dA_t}{d\langle M\rangle_t} = \frac{\delta(u, V_{u-})a + b(u, V_{u-})}{S_{t-}\big(\sigma(u, V_{u-})^2 + \delta(u, V_{u-})^2 \int_{\mathbb{R}} x^2 \nu(dx)\big)}. \tag{3.2}$$

Thus density of the minimal martingale measure $\hat{\mathbb{P}}$ is

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}\left(-\int \hat{\lambda} dM\right)_T = \mathcal{E}\left(\int \hat{H}_u dB_u + \iint_{\mathbb{R}} \hat{h}(u, x) N(du, dx)\right)_T \tag{3.3}$$

where

$$\hat{H}_t = -\hat{\lambda}_t \sigma(t, V_{t-}) S_{t-}, \tag{3.4}$$

$$\text{and} \quad \hat{h}(t, x) = -\hat{\lambda}_t \delta(t, V_{t-}) S_{t-} x. \tag{3.5}$$

We will denote by $\hat{B}_t$ and $\hat{N}(dt, dx)$ the associated transformations of B_t and $N(dt, dx)$ by $\hat{H}_t$ and $\hat{h}(t, x)$, respectively, i.e.

$$d\hat{B}_t = dB_t - \hat{H}_t dt, \tag{3.6}$$

$$\begin{aligned} \hat{N}(dt, dx) &= N(dt, dx) - \hat{h}(t, x)\nu(dx)dt \\ &= J(dt, dx) - \hat{\nu}(dt, dx). \end{aligned} \tag{3.7}$$

3.3 The variance-optimal martingale measure

Let $\mathscr{M}_2^s(\mathbb{P})$ denote the convex set of all signed local martingale measures with square integrable density, ie. $\mathscr{M}_2^s(\mathbb{P}) := \mathscr{M}^s(\mathbb{P}) \cap L^2(\mathbb{P})$. Note that $\mathscr{M}_2^s(\mathbb{P})$ is closed in $L^2(\mathbb{P})$ and has a unique element with minimal $L^2(\mathbb{P})$-norm, due to convexity of the norm (provided $\mathscr{M}_2^s(\mathbb{P}) \neq \varnothing$). Thus we can define the following

Definition 3.2. *Assume that $\mathscr{M}_2^s(\mathbb{P}) \neq \varnothing$. A signed martingale measure $\tilde{\mathbb{P}}$ is called variance-optimal if $\tilde{\mathbb{P}}$ minimizes*

$$\left\| \frac{d\mathbb{Q}}{d\mathbb{P}} \right\|_{L^2(\mathbb{P})} = \left(\mathrm{Var}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) + 1 \right)^{1/2} \qquad \textit{over all} \quad \mathbb{Q} \in \mathscr{M}_2^s.$$

In this section we assume that $G_T(\Theta)$ is a linear subspace of L^2, which corresponds to a frictionless financial market (no transaction costs, taxes, etc.). The assumptions $\mathscr{M}_2^s(\mathbb{P}) \neq \varnothing$ is equivalent to assuming that $1 \notin \overline{G_T(\Theta)}$, cf. Delbaen and Schachermayer (1996c). In the financial context, this can be regarded as a no-arbitrage condition.

Example 3.2. *In this example, we calculate the explicit form of the variance-optimal martingale measure $\tilde{\mathbb{P}}$ for the geometric Lévy model given by the equation* (2.16). *By definition*

$$\begin{aligned}
\left\| \frac{d\mathbb{Q}}{d\mathbb{P}} \right\|^2_{L^2(\mathbb{P})} &= \mathbb{E}[Z_T^2] \\
&= \mathbb{E}\Bigg[\exp\left\{ 2\int_0^T H_s dB_s - \int_0^T H_s^2 ds + 2\int_0^T \int_{\mathbb{R}} h(s,x) N(ds,dx) \right\} \\
&\qquad \times \prod_{0<s\leq T} \big(h(s,\triangle X_s)+1\big)^2 \exp\big(-2h(s,\triangle X_s)\big) \Bigg] \\
&= \mathbb{E}\left[\exp\left\{ \int_0^T H_s^2 ds - 2\int_0^T \int_{\mathbb{R}} h(s,x)\nu(dx)ds \right\} \prod_{0<s\leq T} \big(h(s,\triangle X_s)+1\big)^2 \right] \\
&= \exp\left\{ \int_0^T H_s^2 ds - 2\int_0^T \int_{\mathbb{R}} h(s,x)\nu(dx)ds \right\} \\
&\qquad \times \mathbb{E}\left[\exp\left\{ 2\int_0^T \int_{\mathbb{R}} \ln\big(h(s,x)+1\big) J(ds,dx) \right\} \right],
\end{aligned} \tag{3.8}$$

In the last line we used the fact that H and h are deterministic processes, which is implied by the fact that all model parameters in the martingale condition (2.24) *are* *deterministic themselves. Using Itô's formula, it is clear that*

$$\mathbb{E}\left[\exp\left\{2\int_0^T\int_{\mathbb{R}}\ln\big(h(s,x)+1\big)J(ds,dx)\right\}\right]=\exp\left\{\int_0^T\int_{\mathbb{R}}\big(e^{2\ln(h(s,x)+1)}-1\big)\nu(dx)ds\right\}. \tag{3.9}$$

Putting the results (3.8) *and* (3.9) *together, we have*

$$\left\|\frac{d\mathbb{Q}}{d\mathbb{P}}\right\|^2_{L^2(\mathbb{P})}=\exp\left\{\int_0^T H_s^2ds+\int_0^T\int_{\mathbb{R}}h(s,x)^2\nu(dx)ds\right\}. \tag{3.10}$$

To finish our example, it remains to find the explicit form of the processes H and h that minimize (3.10) *subject to the martingale condition* (2.24)*. We follow the same procedure as in Chan (1999). The most convenient way to solve this optimization problem is the use of Lagrange multipliers. First fix H, let κ be a continuous function and define the Lagrangian*

$$L(\kappa,h)=\int_{\mathbb{R}}h(s,x)^2\nu(dx)+\int_{\mathbb{R}}\kappa_s\sigma_s h(s,x)x\nu(dx). \tag{3.11}$$

Noting that $h\mapsto L(\kappa,h)$ is convex in h, finding the value of h that minimizes (3.11) *requires*

$$\left.\frac{d}{dt}L(\kappa,h+tF)\right|_{t=0}=0$$

for all F, which gives

$$h(s,x)=-\frac{\kappa_s\sigma_s x}{2}.$$

Plugging the optimal value of h into (2.24) *and differentiating, we find that*

$$\kappa_s'(H)=2c\sigma_s\left(\int_{\mathbb{R}}(\sigma_s x)^2\nu(dx)\right)^{-1}. \tag{3.12}$$

Turning on to minimizing H, again using the optimal value of h, we simply differentiate the exponent in the equation (3.10) *which gives*

$$2H_s+\kappa_s'(H)\int_{\mathbb{R}}\frac{\kappa_s(\sigma_s x)^2}{2}\nu(dx)=0,$$

and using (3.12) *we find the optimal value of H to be*

$$H_s=-\frac{\sigma_s\kappa_s c}{2}.$$

For the sake of completeness, after using the optimal forms of H and h in (2.24)*, we have that*

$$\kappa_s=\frac{a\sigma_s+b_s}{\frac{\sigma_s^2}{2}\left[c^2+\int_{\mathbb{R}}x^2\nu(dx)\right]}.$$

Comparing the results with (3.11) of Chan (1999), we see that the minimal and the variance-optimal martingale measure are exactly the same. This is always the case when the mean-variance tradeoff process process is deterministic, cf. Theorem 11 in Schweizer (1995). Another way to see this, is to look at the form of both measures. It was shown in various degrees of generality, that the density of the variance-optimal martingale measure can be written as $c + \int \vartheta dS$ for some $c \in \mathbb{R}$ and $\vartheta \in \Theta$, cf. Delbaen and Schachermayer (1996c) or Schweizer (1996):

Lemma 3.3. *Assume that $\mathscr{M}_2^s(\mathbb{P}) \neq \varnothing$. Then $\tilde{\mathbb{P}} \in \mathscr{M}_2^s$ is variance-optimal, if and only if*

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = [1, \infty) + \overline{G_T(\Theta)}.$$

Proof. cf. Schweizer (1996), proof of Lemma 1, (c). □

In the case of deterministic continuous mean-variance tradeoff

$$\mathcal{E}\left(-\int \hat{\lambda} dM\right) = \mathcal{E}\left(-\int \hat{\lambda} d(S-A)\right) = \mathcal{E}\left(-\int \hat{\lambda} dS\right) \mathcal{E}\left(\hat{K}\right)$$

and thus both measures coincide.

Another characterization of $\tilde{\mathbb{P}}$ is through the so-called adjustment process.

Definition 3.4. *A process $\beta \in L(S)$, the space of S-integrable predictable processes, is called an adjustment process if the following two conditions hold:*

1. $\beta \mathcal{E}\left(-\int \beta dS\right)_- \in \Theta$,

2. $\mathbb{E}[\mathcal{E}\left(-\int \beta dS\right)_T G_T(\vartheta)] = 0$ *for all* $\vartheta \in \Theta$.

Proposition 3.5. *Assume that $\mathscr{M}_2^s(\mathbb{P}) \neq \varnothing$. If β is an adjustment process, then $\tilde{\mathbb{P}}$ is given by*

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{\mathcal{E}\left(-\int \beta dS\right)_T}{\mathbb{E}[\mathcal{E}\left(-\int \beta dS\right)_T]}.$$

Proof. cf. Schweizer (1996), proof of Proposition 8. □

We now proceed to find $\tilde{\mathbb{P}}$ for the stochastic volatility model (2.17) by using the above characterization. We assume the usual framework, a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual conditions generated by B, J and L. Further, we assume that S is locally in $L^2(\mathbb{P})$ in the following sense: there exists a sequence $(\tau_n)_{n=1}^{\infty}$ of localising stopping times increasing to infinity such that, for each $n \in \mathbb{N}$, the family $\{S_T : T \text{ stopping time}, T \le \tau_n\}$ is bounded in $L^2(\mathbb{P})$, cf. Delbaen and Schachermayer (1996a).

Recall the form of $\hat{H}$ and $\hat{h}$ characterizing the minimal martingale measure, given by equations (3.4) and (3.5), respectively. For the rest of the section, we assume the following holds

Assumption 3.1. *The process* $\iint_{\mathbb{R}} |\hat{h}(s,x)| J(ds,dx)$ *is locally* $\mathbb{P}$*-integrable.*

The following is a generalization of Lemma 2.15 in Biagini et al. (2000).

Proposition 3.6. *Let C be a positive constant, and let F and f be predictable. The following conditions are equivalent:*

a)

$$\mathcal{E}\left(\int F_u dB_u^V + \iint_{\mathbb{R}} f(u,x) N^V(du,dx)\right)_T \exp(\hat{K}_T) = C \tag{3.13}$$

b)

$$\begin{aligned} \mathcal{E}\Bigg(\int \hat{H}_u dB_u + \iint_{\mathbb{R}} \hat{h}(u,x) N(du,dx) + \\ \int F_s dB_u^V + \iint_{\mathbb{R}} f(u,x) N^V(ds,dx)\Bigg)_T = \\ C\mathcal{E}\left(\int \hat{H}_u d\hat{B}_u + \iint_{\mathbb{R}} \hat{h}(u,x)\hat{N}(du,dx)\right)_T. \end{aligned} \tag{3.14}$$

Proof. This follows from equations (3.1), (3.4), (3.5), (3.6) and (3.7). □

Remark 3.1. Equation (3.13) is a generalization of the *representation equation* derived by Biagini et al. (2000) and Hobson (2004) for diffusion stochastic volatility models and by Rheinländer (2005) for the Stein and Stein model. Here the representation equation is more involved, which reflects the fact that we are dealing with a volatility process and a price process that are discontinuous.

We now move onto specifying the set of admissible strategies that we are going to use. A process ϑ is called a simple trading strategy if it has a form $\vartheta = h1_{(\tau_1,\tau_2]}$ where $\tau_1 < \tau_2$ denote stopping times and h is bounded and $\mathcal{F}_{\tau_1}$ measurable.

To guarantee $L^2(\mathbb{P})$ closedness, we assume the following space of admissible strategies, see Delbaen and Schachermayer (1996c), Gourieroux, Laurent and Pham (1998) and the subsequent formulation of Černý and Kallsen (2007) and Xia and Yan (2006) that we use here.

Definition 3.7. *A strategy $\vartheta \in L(S)$ is admissible, if there exists a sequence $(\vartheta^{(n)})_{n\in\mathbb{N}}$ of simple trading strategies such that*

$$\int_0^T \vartheta dS = L^2 - \lim_{n\to\infty} \int_0^T \vartheta^n dS, \int_0^t \vartheta dS = \lim_{n\to\infty} \int_0^t \vartheta^n dS \quad \textit{in probability for all } t.$$

The space of admissible strategies is denoted by Θ.

The following result provides characterisation of admissible strategies.

Corollary 3.8. *We have equivalence between:*
1. ϑ is an admissible strategy
2. $\vartheta \in L(S)$, $\int \vartheta dS \in L^2(\mathbb{P})$, and $Z^{\mathbb{Q}} \int \vartheta dS$ is a martingale for any sigma martingale measure $\mathbb{Q}$ with density process $Z^{\mathbb{Q}}$ and $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^2(\mathbb{P})$.

Proof. cf. Černý and Kallsen (2007), proof of Corollary 2.5. □

We also introduce alternative space of admissible strategies. For any special semimartingale Y with Doob-Meyer decomposition $Y = Y_0 + M^Y + A^Y$, with $M_0^Y = A_0^Y = 0$, define

$$\| Y \|_{\mathscr{H}^2} = \| ([M^Y, M^Y]_T)^{\frac{1}{2}} \|_{L^2(\mathbb{P})} + \| \mathrm{Var}(A^Y)_T \|_{L^2(\mathbb{P})}, \tag{3.15}$$

where $\mathrm{Var}(A^Y)$ denotes variation of A^Y. Then Y belongs to a set of square integrable semimartingales $\mathscr{H}^2$ if $\| Y \|_{\mathscr{H}^2} < \infty$. We denote by Θ^H the set of following strategies

$$\Theta^H := \{\vartheta \in L(S) : \int \vartheta dS \in \mathscr{H}^2\}.$$

It is in general easier to check whether certain strategy belongs to Θ^H, on the other hand, as opposed to $G_T(\Theta)$, the set $G_T(\Theta^H)$ is not necessarily closed. In this respect, a useful result is provided by Corollary 2.9 of Černý and Kallsen (2007) which shows that $\Theta^H \subset \Theta$.

Remark 3.2. In our stochastic volatility model we have

$$\begin{aligned}
\| \int_0^T \hat{\lambda} dS \|_{\mathscr{H}^2} &= \| (\int_0^T \hat{\lambda}^2 d[M,M])^{\frac{1}{2}} \|_{L^2(\mathbb{P})} + \| \int_0^T |\hat{\lambda}||dA| \|_{L^2(\mathbb{P})} \\
&= \mathbb{E}[\int_0^T \hat{\lambda}^2 d[M,M]]^{\frac{1}{2}} + \mathbb{E}[(\int_0^T \hat{\lambda} dA)^2]^{\frac{1}{2}} \\
&= \mathbb{E}[\int_0^T \hat{\lambda}^2 d\langle M,M\rangle]^{\frac{1}{2}} + \mathbb{E}[\hat{K}_T^2]^{\frac{1}{2}} \\
&= \mathbb{E}[\hat{K}_T]^{\frac{1}{2}} + \mathbb{E}[\hat{K}_T^2]^{\frac{1}{2}}.
\end{aligned}$$

Last line implies that $\hat{\lambda} \in \Theta^H$ if and only if the mean-variance tradeoff process $\hat{K}$ has finite second moment.

We can now use the Proposition 3.6 to obtain the candidate variance-optimal martingale measure.

Lemma 3.9. *Let $\mathbb{Q}$ be the signed local martingale measure given by*

$$\begin{aligned}
\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}\Bigg(\int \hat{H}_u dB_u &+ \iint_{\mathbb{R}} \hat{h}(u,x) N(du,dx) \\
&+ \int F_s dB_u^V + \iint_{\mathbb{R}} f(u,x) N^V(ds,dx) \Bigg)_T
\end{aligned}$$

for processes F and f. Then the density of $\mathbb{Q}$ with respect to $\mathbb{P}$ is of the form

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\mathcal{E}(-\int \hat{\lambda} dS)_T}{\mathbb{E}\left[\mathcal{E}(-\int \hat{\lambda} dS)_T\right]} \tag{3.16}$$

if and only if the representation equation (3.13) *holds with C given by*

$$\begin{aligned}
C &= \mathbb{E}\left[\mathcal{E}(-\textstyle\int \hat{\lambda} dS)_T\right]^{-1} \\
&= \mathcal{E}\left(\int F_u dB_u^V + \iint_{\mathbb{R}} f(u,x) N^V(du,dx) \right)_T \exp(\hat{K}_T).
\end{aligned} \tag{3.17}$$

Proof. First assume that F and f are given by the representation equation (3.13). Note that the process S can be written as

$$S_t = S_0 + \int_0^t S_{u-} \left(\sigma(u, V_{u-}) d\hat{B}_u + \delta(u, V_{u-}) \int_{\mathbb{R}} x \hat{N}(du,dx) \right).$$

The form of $\hat{H}$ and most importantly the form of $\hat{h}(t,x)$, see equations (3.4) and

(3.5), imply that

$$\begin{aligned}
&\int_0^T \hat{H}_u d\hat{B}_u + \int_0^T\int_{\mathbb{R}} \hat{h}(u,x)\hat{N}(du,dx) \\
&\qquad = -\int_0^T S_{u-}\hat{\lambda}_u \left(\sigma(u,V_{u-})d\hat{B}_u + \delta(u,V_{u-})\int_{\mathbb{R}} x\hat{N}(du,dx)\right) \\
&\qquad = -\int_0^T \hat{\lambda}_u dS_u.
\end{aligned}$$

This in turn implies

$$\begin{aligned}
\frac{d\mathbb{Q}}{d\mathbb{P}} &= \mathcal{E}\left(-\int \hat{\lambda}_u dS_u\right)_T \\
&\quad \times \mathcal{E}\left(\int F_u dB_u^V + \iint_{\mathbb{R}} f(u,x)N^V(du,dx)\right)_T \exp(\hat{K}_T),
\end{aligned} \tag{3.18}$$

and the representation equation (3.13) yields

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\mathcal{E}\left(-\int \hat{\lambda}_u dS_u\right)_T}{\mathbb{E}\left[\mathcal{E}\left(-\int \hat{\lambda}_u dS_u\right)_T\right]}.$$

The proof in the other direction follows from equation (3.18) above. □

We can now state the main theorem of this section.

Theorem 3.10. *Assume that F and f satisfy equation* (3.17) *and that local martingale Z defined via*

$$Z_t = \mathcal{E}\left(\int \hat{H}_u dB_u + \iint_{\mathbb{R}} \hat{h}(u,x)N(du,dx) + \int F_s dB_u^V + \iint_{\mathbb{R}} f(u,x)N^V(ds,dx)\right)_t$$

is a true martingale. Further assume that $\mathcal{E}\left(-\int \hat{\lambda} dS\right)_T$ is square-integrable, that $\mathcal{E}\left(-\int \hat{\lambda} dM\right)$ is integrable and that $\mathbb{E}[\exp(-\hat{K}_T)] > 0$ $\mathbb{P}$-a.s.. Then the measure $\mathbb{Q}$ defined via $d\mathbb{Q}/d\mathbb{P} = Z_T$ is the variance-optimal signed martingale measure.

Proof. By Lemma 3.9, the measure $\mathbb{Q}$ takes the form of the measure $\tilde{\mathbb{P}}$ from Proposition 3.5 with $\beta = \hat{\lambda}$. The prove that $\mathbb{Q}$ is variance-optimal, we need to show that $\hat{\lambda}$ is an adjustment process and that $\mathscr{M}_2^s(\mathbb{P}) \neq \varnothing$. We start by showing that the second property of the adjustment process, c.f. Definition 3.4, holds. For any $\vartheta \in \Theta$ we have

$$\mathbb{E}\left[\mathcal{E}\left(-\int \hat{\lambda} dS\right)_T G_T(\vartheta)\right] = \mathbb{E}\left[\mathcal{E}\left(-\int \hat{\lambda} dM\right)_T \exp(-\hat{K}_T)G_T(\vartheta)\right] \tag{3.19}$$

because A is continuous and finite variation process. The expectation is finite due to square-integrability assumption of $\mathcal{E}\left(-\int \hat{\lambda} dS\right)_T$ and the fact that $\vartheta \in \Theta$, using Cauchy-Schwarz inequality. Now define a martingale $M_t^V = \mathbb{E}[\exp(-\hat{K}_T)|\mathcal{F}_t]$. As the process $\hat{K}$ is independent of S, by the martingale representation theorem, cf. Theorem 2.6, we have

$$M_t^V = \mathbb{E}[\exp(-\hat{K}_T)] + \int_0^t \phi_s dB_s^V + \int_0^t \int_{\mathbb{R}} \psi(s,x) N^V(ds,dx),$$

for some processes (ϕ, ψ), where integrability of M^V is guaranteed by the fact that $\mathbb{E}[\exp(-\hat{K}_T)|\mathcal{F}_t] \leq 1$. Because M is orthogonal to M^V, we have

$$\begin{aligned}\mathcal{E}\left(-\int \hat{\lambda} dM\right)_T M_T^V = \mathbb{E}[\exp(-\hat{K}_T)] &- \int_0^T M_t^V \hat{\lambda}_t \mathcal{E}\left(-\int \hat{\lambda} dM\right)_{t-} dM_t \\ &+ \int_0^T \mathcal{E}\left(-\int \hat{\lambda} dM\right)_t \left(\phi_t dB_t^V + \int_{\mathbb{R}} \psi(t,x) N^V(dt,dx)\right)\end{aligned}$$

which is by Proposition 2.18 a martingale density for S, after scaling by $\mathbb{E}[\exp(-\hat{K}_T)]$. Note that $\mathcal{E}\left(-\int \hat{\lambda} dM\right)$ is a true martingale, as both Z and M^V are true martingales. Thus the integrability of the product above follows from

$$\mathbb{E}\left[|\mathcal{E}\left(-\int \hat{\lambda} dM\right)_t M_t^V|\right] \leq \mathbb{E}\left[|\mathcal{E}\left(-\int \hat{\lambda} dM\right)_t|\right] < \infty.$$

Therefore

$$\begin{aligned}(3.19) &= \mathbb{E}\left[\mathcal{E}\left(-\int \hat{\lambda} dM\right)_T M_T^V G_T(\vartheta)\right] \\ &= 0,\end{aligned}$$

which proves the second property of the adjustment process.

We now follow the arguments of the proof of Theorem 3.3.2 in Wiese (1998) to show that the first property of the adjustment process holds. Let $\bar{Z}$ be the solution to $\bar{Z} = 1 - \int \hat{\lambda} \bar{Z}_- dS$. For $n \in \mathbb{N}$, define $T_n := \inf\{t > 0 : \hat{K}_t \geq n\}$. Consider the stopped process S^{T_n}. By Protter (2004), Theorem 2.18, the stopped process $\bar{Z}^{T_n}$ satisfies

$$\bar{Z}^{T_n} = 1 - \int \hat{\lambda} \bar{Z}_- dS^{T_n} = 1 - \int \hat{\lambda} 1_{[0,T_n]} \bar{Z}_-^{T_n} dS.$$

In the first step we show that the integrand, $\hat{\lambda}1_{[0,T_n]}\bar{Z}_-^{T_n}$, is admissible. We have

$$
\begin{aligned}
\| \bar{Z}^{T_n} \|_{\mathscr{H}^2} &= \| \hat{Z}^{T_n} \exp(-\hat{K}^{T_n}) \|_{\mathscr{H}^2} \leq \| \hat{Z}^{T_n} \|_{\mathscr{H}^2} \\
&= \| \Big(\int_0^T \hat{\lambda}^2 1_{[0,T_n]} (\hat{Z}_-^{T_n})^2 d[M,M] \Big)^{\frac{1}{2}} \|_{L^2(\mathbb{P})} \\
&= \mathbb{E}\Big[\int_0^T \hat{\lambda}^2 1_{[0,T_n]} (\hat{Z}_-^{T_n})^2 d[M,M] \Big]^{\frac{1}{2}} \\
&= \mathbb{E}\Big[\int_0^T \hat{\lambda}^2 1_{[0,T_n]} (\hat{Z}_-^{T_n})^2 d\langle M,M\rangle \Big]^{\frac{1}{2}} \\
&\leq \mathbb{E}\Big[\sup_{t\in[0,T]} (\hat{Z}_{t-}^{T_n})^2 \int_0^T \hat{\lambda}^2 1_{[0,T_n]} d\langle M,M\rangle \Big]^{\frac{1}{2}} \\
&= \mathbb{E}\Big[\sup_{t\in[0,T]} (\hat{Z}_{t-}^{T_n})^2 \hat{K}_T^{T_n} \Big]^{\frac{1}{2}} = \sqrt{n}\,\mathbb{E}\Big[\sup_{t\in[0,T]} (\hat{Z}_{t-}^{T_n})^2 \Big]^{\frac{1}{2}} \\
&\leq 2\sqrt{n}\,\mathbb{E}[(\hat{Z}_T^{T_n})^2]^{\frac{1}{2}} \leq 2\sqrt{n}\,\mathbb{E}[(\bar{Z}_T^{T_n})^2 \exp(2\hat{K}_T^{T_n})]^{\frac{1}{2}} && (3.20) \\
&= 2\sqrt{n}\exp(n)\mathbb{E}[\bar{Z}_T^{T_n}]^{\frac{1}{2}} = 2\sqrt{n}\exp(n)\mathbb{E}[\hat{Z}_T^{T_n}\exp(-\hat{K}_T^{T_n})]^{\frac{1}{2}} && (3.21) \\
&= 2\sqrt{n}\exp(n/2). && (3.22)
\end{aligned}
$$

Line (3.20) follows from Doob's maximal quadratic inequality, where we used the fact that $\hat{Z}$ is a martingale by assumption of the Theorem. Line (3.21) is justified by using the second property of the adjustment process, which we have already proved to hold for time T, and the fact that it holds also for the stopped process S^{T_n}, due to independence of the stopping time T_n from S, i.e. $\mathbb{E}[(\bar{Z}_T^{T_n})^2] = \mathbb{E}[\bar{Z}_T^{T_n}]$. Finally, (3.22) follows from the definition of the stopping time T_n and the fact that $\hat{Z}$ is a martingale. Thus we have shown that $\hat{\lambda}1_{(0,T_n]}\bar{Z}_-^{T_n} \in \Theta^H$ and therefore admissible by Corollary 2.9 of Černý and Kallsen (2007). As both properties of the adjustment process hold for T_n, $\hat{\lambda}1_{(0,T_n]}$ is the adjustment process for the stopped process S^{T_n} and the density of the variance-optimal martingale measure for the stopped process S^{T_n} is given by $\bar{Z}_T^{T_n}/\mathbb{E}[\bar{Z}_T^{T_n}]$. Thus $\bar{Z}_T^{T_n} \in L^2(\mathbb{P})$ for $n \in \mathbb{N}$, $\{\bar{Z}_T^{T_n}, n \in \mathbb{N}\}$ is uniformly integrable in $L^2(\mathbb{P})$ and therefore $\mathbb{E}[\bar{Z}_T^{T_n}] \to \mathbb{E}[\bar{Z}_T]$, cf. Gut (2005), Theorems 5.4.2 and 5.5.2. As $\hat{\lambda}1_{(0,T_n]}$ is the adjustment process for the stopped process S^{T_n} for $n \in \mathbb{N}$, we have that $\mathbb{E}[\bar{Z}_T^{T_n}]=\mathbb{E}[(\bar{Z}_T^{T_n})^2]$, and similarly, as the second property of the adjustment process was shown to hold in the first part of the proof, we have that $\mathbb{E}[\bar{Z}_T]=\mathbb{E}[(\bar{Z}_T)^2]$. Thus $\mathbb{E}[(\bar{Z}_T^{T_n})^2] \to \mathbb{E}[(\bar{Z}_T)^2]$ which by Gut (2005), Theorem 5.5.2 implies that $\int_0^T \hat{\lambda}1_{(0,T_n]}\bar{Z}_-^{T_n} dS$ converges to $\int_0^T \hat{\lambda}\bar{Z}_- dS$ in $L^2(\mathbb{P})$. This together with the admissibility of $\hat{\lambda}1_{(0,T_n]}\bar{Z}_-^{T_n}$ now implies that there exists a sequence of simple trading strategies $(\vartheta^n)_{n\in\mathbb{N}}$ such that $\int_0^T \vartheta^n dS$ converges to $\int_0^T \hat{\lambda}\bar{Z}_- dS$ in $L^2(\mathbb{P})$. Note that all results in the proof so far also hold when T is replaced with any positive $t < T$. Thus we also have that $\int_0^t \vartheta^n dS$ converges to $\int_0^t \hat{\lambda}\bar{Z}_- dS$ in $L^2(\mathbb{P})$ and therefore also in probability. This shows that $\hat{\lambda}\bar{Z}_-$ is admissible.

Combining the results above imply that $\hat{\lambda}$ is an adjustment process. It only remains to show that $\bar{Z}_T/\mathbb{E}[\bar{Z}_T]$ has finite second moment. From equation (3.17) we have that $\mathbb{E}[\bar{Z}_T] = \mathbb{E}[\exp(-\hat{K}_T)]$. As $\hat{\lambda}$ is an adjustment process, $\mathbb{E}[\bar{Z}_T] = \mathbb{E}[\bar{Z}_T^2]$. Thus

$$\mathbb{E}\left[\left(\frac{\bar{Z}_T}{\mathbb{E}[\bar{Z}_T]}\right)^2\right] = \frac{\mathbb{E}[\bar{Z}_T^2]}{\mathbb{E}[\bar{Z}_T]^2} = \frac{\mathbb{E}[\bar{Z}_T]}{\mathbb{E}[\bar{Z}_T]^2} = \frac{1}{\mathbb{E}[\exp(-\hat{K}_T)]},$$

which is finite by assumption and therefore $\bar{Z}_T/\mathbb{E}[\bar{Z}_T]$ is the density of the variance-optimal martingale measure. □

3.3.1 Černý-Kallsen approach

We shortly comment on a different approach to determine the variance-optimal martingale measure. Černý and Kallsen (2007) suggest an approach based on finding a measure $\mathbb{P}^*$ that neutralizes the effect of the stochastic mean-variance tradeoff process, and the variance-optimal martingale measure is then computed as the minimal martingale measure with respect to $\mathbb{P}^*$. Černý and Kallsen call $\mathbb{P}^*$ the opportunity neutral measure, see Lemma 3.15 and Definition 3.16. The above can be written as

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{d\mathbb{P}^*}{d\mathbb{P}}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*} = \frac{d\mathbb{P}^*}{d\mathbb{P}}\frac{d\widehat{\mathbb{P}^*}}{d\mathbb{P}^*}.$$

In comparison, the approach used here is to determine the minimal martingale measure first, and in the second step the representation equation is used to determine $\tilde{\mathbb{P}}$,

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\frac{d\tilde{\mathbb{P}}}{d\hat{\mathbb{P}}}.$$

However in the uncorrelated case, the equation (3.27) in Černý and Kallsen (2007) that is used to characterize the opportunity process L and which in turn defines $\mathbb{P}^*$, simplifies considerably. In our model, the equation (3.27) means that the drift process of the stochastic logarithm of L under $\mathbb{P}$ is equal to the mean-variance tradeoff process $\hat{K}$, and going from measure $\mathbb{P}$ to $\mathbb{P}^*$ will not change the drift of S (note that $\mathbb{P}^*$ is not a martingale measure). But this implies that

$$\frac{d\widehat{\mathbb{P}^*}}{d\mathbb{P}^*} = \frac{d\hat{\mathbb{P}}}{d\mathbb{P}} \quad \text{and} \quad \frac{d\mathbb{P}^*}{d\mathbb{P}} = \frac{d\tilde{\mathbb{P}}}{d\hat{\mathbb{P}}},$$

ie. finding $\tilde{\mathbb{P}}$ through the measure $\mathbb{P}^*$ is equivalent to finding $\tilde{\mathbb{P}}$ by using the representation equation, and the process L can be recovered from the following expression

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \frac{L}{\mathbb{E}[L_0]\mathcal{E}(\hat{K})}.$$

3.4 The minimal entropy martingale measure

The minimal entropy martingale measure is well studied subject in the mathematical finance and particularly in the context of Lévy processes, cf. Chan (1999), Frittelli (2000), Fujiwara and Miyahara (2003), Grandits and Rheinländer (2002), Rheinländer (1999), Rheinländer (2005), Esche and Schweizer (2005), Rheinländer and Steiger (2006). It has been used for pricing purposes as a martingale measure with minimal relative entropy with respect to the historical probability or for calibration purposes, see for example Carr et al. (2002), Tankov (2004).

While the relative entropy is not a metric, it can be viewed as a distance between two probability measures. The relative entropy, also called Kullback–Leibler divergence, $I(\mathbb{Q},\mathbb{P})$ of a probability measure $\mathbb{Q}$ with respect to a probability measure $\mathbb{P}$ is given as

$$I(\mathbb{Q},\mathbb{P}) = \begin{cases} \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\log\frac{d\mathbb{Q}}{d\mathbb{P}}\right] & \text{if } \mathbb{Q} \ll \mathbb{P}, \\ +\infty & \text{otherwise.} \end{cases}$$

Note that $I(\mathbb{Q},\mathbb{P}) \geq 0$ for any probability measure $\mathbb{Q}$ and $I(\mathbb{Q},\mathbb{P}) = 0$ if and only if $\mathbb{Q} = \mathbb{P}$. The functional $I(\mathbb{Q},\mathbb{P})$ is strictly convex in the first argument. A measure $\mathbb{Q}^{(e)} \in \mathscr{M}(\mathbb{P})$ which satisfies

$$I(\mathbb{Q}^{(e)},\mathbb{P}) = \min_{\mathbb{Q}\in\mathscr{M}(\mathbb{P})} I(\mathbb{Q},\mathbb{P})$$

is called the *minimal entropy martingale measure*.

In the exponential/geometric Lévy models, the minimal entropy martingale measure preserves Lévy property. This can be seen in Chan (1999) and has been rigorously proved in Esche and Schweizer (2005). The connection between $\mathbb{Q}^{(e)}$ and Esscher transform has been shown in Chan (1999) and further generalized and studied in Fujiwara and Miyahara (2003) and Hubalek and Sgarra (2006).

Frittelli (2000) proves in Theorem 2.1 that in the case of bounded processes, the sufficient condition for existence of the minimal entropy martingale measure is existence of a martingale measure with finite relative entropy with respect to $\mathbb{P}$. Moreover, if there exists an equivalent martingale measure with finite relative entropy, then the minimal entropy martingale measure is equivalent. But in the case of unbounded processes, $\inf_{\mathbb{Q}\in\mathscr{M}(\mathbb{P})} I(\mathbb{Q},\mathbb{P})$ may not be attained by a martingale measure.

For the remainder of this section, we assume that following holds

Assumption 3.2. *S is a locally bounded.*

By Proposition 3.2 of Grandits and Rheinländer (2002), the density of the minimal entropy martingale measure $\mathbb{Q}^{(e)}$ is necessary of the form

$$\frac{d\mathbb{Q}^{(e)}}{d\mathbb{P}} = \exp\left(c - \int_0^T \lambda_t dS_t\right) \tag{3.23}$$

where c is a constant and λ is a predictable process such that $\int \lambda dS$ is a $\mathbb{Q}^{(e)}$-martingale. That (3.23) is not sufficient to characterize the martingale measure with minimal entropy has been shown by counterexample in Schachermayer (2003). Thus after finding a candidate measure $\mathbb{Q}^{(c)}$, which we define to be any martingale measure of the form (3.23), one needs to verify that the candidate measure is entropy minimizer.

Let us start by finding the candidate martingale measure. Comparing (2.25) and (3.23), we see that the candidate minimal entropy martingale measure corresponds to the choices

$$H_t = -\lambda_t S_{t-}\sigma_t, \tag{3.24}$$

$$\text{and } h(t,x) = \exp(-\lambda_t S_{t-}\delta_t x) - 1. \tag{3.25}$$

Therefore, we define the following mapping (dependence on V_{t-} omitted from notation)

$$\Psi(\lambda_t,t) = -\lambda_t S_{t-}\sigma_t^2 + b_t + \delta_t \mathbb{E}(J_1) + \delta_t \int_{\mathbb{R}} x(e^{-\lambda_t S_{t-}\delta_t x} - 1)\nu(dx) \tag{3.26}$$

and assume that the process λ is such that

$$\int_{\mathbb{R}} |x(e^{-\lambda_t S_{t-}\delta_t x} - 1)|\nu(dx) < \infty.$$

This guarantees that the integral in (3.26) is well defined. This will ultimately depend on how heavy are the tails of the Lévy measure. Further assume that for any $t \in [0,T]$, λ is a solution to $\Psi(\lambda_t,t) = 0$. Note that when such λ exists, it is unique as $\Psi(\lambda_t,t)$ is monotonically decreasing as a function of λ_t in its domain, irrespective of the sign of δ. It is now a straightforward calculation to show that a local martingale measure $\mathbb{Q}^{(c)}$ is of the form (3.23) if and only if F and f defined in (2.25) satisfy

$$\mathcal{E}\left(\int F_u dB_u^V + \int\int f(u,x) N^V(du,dx)\right)_T = \\ \exp\left(c - \frac{1}{2}\int_0^T (\sigma_u \lambda_u S_{u-})^2 du - \int_0^T \int_{\mathbb{R}} \left(e^{-\lambda_u S_{u-}\delta_u x}(-\lambda_u S_{u-}\delta_u x - 1) + 1\right)\nu(du,dx)\right). \tag{3.27}$$

The condition that guarantees the existence and uniqueness of the solution to the equation $\Psi(\lambda_t, t) = 0$ for all t is given in the following

Proposition 3.11. *Assume that J satisfies the following condition*

$$\int_{\{|x|\geq 1\}} e^{ux}\nu(dx) < \infty, \quad \text{for any } u \in (-u_1, u_2), \text{ where } 0 < u_1, u_2 \leq \infty.$$

Then the equation $\Psi(\lambda_t, t) = 0$ has a unique solution for all t. If furthermore $b_t + \delta_t \mathbb{E}(J_1) > 0$, then $\lambda_t \in \left(0, \dfrac{b_t + \delta_t \mathbb{E}(J_1)}{S_{t-}\sigma_t^2}\right)$.

Proof. It follows from the assumption that

$$\int_{\{|x|\geq 1\}} x^\gamma e^{ux}\nu(dx) < \infty \qquad \forall \gamma > 0.$$

which implies that the integral in the definition of $\Psi(\lambda_t, t)$ is finite for $\lambda_t \in (-u_2/(S_{t-}\delta_t), u_1/(S_{t-}\delta_t))$ when $\delta_t > 0$ and for $\lambda_t \in (u_1/(S_{t-}\delta_t), -u_2/(S_{t-}\delta_t))$ when $\delta_t < 0$, and thus $\Psi(\lambda_t, t)$ is well defined. For a fixed t, $\Psi(\lambda_t, t)$ is monotonically decreasing as a function of λ_t (irrespective of the sign of δ_t). Moreover,

$$\lim_{\lambda \downarrow -\frac{u_2}{S_{t-}\delta_t}} \Psi(\lambda, t) = \infty \quad \text{and} \quad \lim_{\lambda \uparrow \frac{u_1}{S_{t-}\delta_t}} \Psi(\lambda, t) = -\infty \qquad \text{for } \delta_t > 0,$$

$$\lim_{\lambda \downarrow \frac{u_1}{S_{t-}\delta_t}} \Psi(\lambda, t) = \infty \quad \text{and} \quad \lim_{\lambda \uparrow \frac{-u_2}{S_{t-}\delta_t}} \Psi(\lambda, t) = -\infty \qquad \text{for } \delta_t < 0.$$

Thus equations of the form $\Psi(\lambda, \cdot) = const$ have unique solution for all λ for which $\Psi(\lambda, \cdot) < \infty$.

For the second assertion, note that $\Psi(0, t) = b_t + \delta_t \mathbb{E}(J_1)$ and for $\lambda_t = \dfrac{b_t + \delta_t \mathbb{E}(J_1)}{S_{t-}\sigma_t^2}$, only the integral part in the expression of $\Psi(q, \lambda, t)$ is non-zero. However, because the integral part is monotonically decreasing in λ_t (irrespective of the sign of δ_t) and it is zero at $\lambda_t = 0$, the integral part and thus $\Psi(q, \lambda, t)$ must be negative at $\lambda_t = \dfrac{b_t + \delta_t \mathbb{E}(J_1)}{S_{t-}\sigma_t^2}$ because of the assumption $b_t + \delta_t \mathbb{E}(J_1) > 0$. This implies that λ_t lies in the interval $\left(0, \dfrac{b_t + \delta_t \mathbb{E}(J_1)}{S_{t-}\sigma_t^2}\right)$. □

The following is the Proposition 3.2, Grandits and Rheinländer (2002), which ensures that the candidate measure $\mathbb{Q}^{(c)}$ minimizes the entropy.

Proposition 3.12. *Assume there exists $\mathbb{Q}^{(c)} \in \mathscr{M}^e(\mathbb{P})$ such that $H(\mathbb{Q}^{(c)}, \mathbb{P}) < \infty$. Then $\mathbb{Q}^{(c)} = \mathbb{Q}^{(e)}$ if and only if the following hold:*

(i) $d\mathbb{Q}^{(c)}/d\mathbb{P} = c\exp((-\int \lambda dS)_T)$ *for a constant c and an S-integrable λ;*

(ii) $\mathbb{E}^{\mathbb{Q}}[(-\int \lambda dS)_T] = 0$ *for* $\mathbb{Q} = \mathbb{Q}^{(c)}, \mathbb{Q}^{(e)}$.

Obviously

$$\begin{aligned} I(\mathbb{Q}^{(e)}, \mathbb{P}) &= \log \mathbb{E}\left[\exp\left(-\frac{1}{2}\int (\sigma_u \lambda_u)^2 du - \int\int \left(e^{-\lambda_u \delta_u x}(-\lambda_u \delta_u x - 1) + 1\right)\nu(du, dx)\right)\right] \\ &= c. \end{aligned}$$

For Lévy processes, we know that the stochastic exponential of a martingale is in fact a martingale, cf. Proposition 1.4 in Tankov (2004). But in general it is only a local martingale. As the minimal entropy martingale measure is a probability measure, one can use Lemma 2.12 to guarantee that $\mathbb{Q}^{(c)}$ is a true martingale measure. When S is locally bounded, a sufficient condition for $\int \lambda dS$ to be a $\mathbb{Q}^{(e)}$-martingale is provided by Proposition 3.2 of Rheinländer (2005). In our case this condition translates to

$$\mathbb{E}\left[\exp\left(\int_0^T \lambda_t^2 S_{t-}^2 \left[(\sigma(t, V_{t-})^2 dt + \delta(t, V_{t-})^2 \int_{\mathbb{R}} x^2 J(dx, dt)\right]\right)\right] < \infty. \tag{3.28}$$

This implies that $\int \lambda dS$ is a true $\mathbb{Q}$-martingale for all $\mathbb{Q} \in \mathscr{M}^e(\mathbb{P})$, and thus $\mathbb{Q}^{(c)} = \mathbb{Q}^{(e)}$ by Proposition 3.12. Note that (3.28) involves the unknown λ. From the second statement of Proposition 3.11, we have upper bound for λ^2 and thus we have the following alternative sufficient condition to (3.28), that involves only model parameters:

$$\mathbb{E}\left[\exp\left(\int_0^T \frac{(b_t + \delta_t \mathbb{E}(J_1))^2}{\sigma(t, V_{t-})^4} \left[(\sigma(t, V_{t-})^2 dt + \delta(t, V_{t-})^2 \int_{\mathbb{R}} x^2 J(dx, dt)\right]\right)\right] < \infty. \tag{3.29}$$

The Proposition 3.12 assumes that there exists equivalent martingale measure with finite entropy. By appealing to the result of Frittelli (2000), Theorem 2.1, we can provide a condition which guarantees that there exists a unique minimal entropy martingale measure. It is sufficient, for example, if the minimal martingale measure has finite entropy. The entropy of the minimal martingale measure is given by

$$
\begin{aligned}
&\mathbb{E}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\log\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right] = \hat{\mathbb{E}}\left[\log\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right] \\
&= \hat{\mathbb{E}}\Bigg[\int_0^T \hat{H}_u d\hat{B}_u + \frac{1}{2}\int_0^T \hat{H}_u^2 du + \int_0^T\int_{\mathbb{R}} \hat{h}(u,x)\hat{N}(du,dx) \\
&\qquad + \int_0^T\int_{\mathbb{R}} \hat{h}(u,x)^2\nu(dx)du + \int_0^T\int_{\mathbb{R}} \big(\log(\hat{h}(u,x)+1) - \hat{h}(u,x)\big)\hat{\nu}(du,dx)\Bigg] \\
&= \mathbb{E}\left[\frac{1}{2}\int_0^T \hat{H}_u^2 du + \int_0^T\int_{\mathbb{R}} \big(\log(\hat{h}(u,x)+1) - \hat{h}(u,x)\big)\big(\hat{h}(u,x)+1\big)\nu(dx)du\right].
\end{aligned}
$$

The last line follows from the fact that the model parameters depend only on the volatility process, which has the same distribution under both measure $\mathbb{P}$ and the minimal martingale measure $\hat{\mathbb{P}}$. Therefore, if

$$
\mathbb{E}\left[\frac{1}{2}\int_0^T \hat{H}_u^2 du + \int_0^T\int_{\mathbb{R}} \big(\log(\hat{h}(u,x)+1) - \hat{h}(u,x)\big)\big(\hat{h}(u,x)+1\big)\nu(dx)du\right] < \infty,
$$

the minimal entropy martingale measure exists.

3.5 The q-optimal martingale measure

In this section we consider a set of signed martingale measures of which the variance-optimal martingale measure is a special case. Let $q > 1$ and p be its conjugate, namely $\frac{1}{p} + \frac{1}{q} = 1$. Similarly to the previous section, we denote by $\mathscr{M}_q^s(\mathbb{P}) := \mathscr{M}^s(\mathbb{P}) \cap L^q(\mathbb{P})$ and $\mathscr{M}_q^e(\mathbb{P}) := \mathscr{M}^e(\mathbb{P}) \cap L^q(\mathbb{P})$. Note that $\mathscr{M}_q^s(\mathbb{P})$ is closed in $L^q(\mathbb{P})$ and has a unique element with minimal $L^q(\mathbb{P})$-norm, due to convexity of the norm (provided $\mathscr{M}_q^s(\mathbb{P}) \neq \varnothing$). Thus we can define the following

Definition 3.13. *Assume that $\mathscr{M}_q^s(\mathbb{P}) \neq \varnothing$. A signed martingale measure $\mathbb{Q}^{(q)}$ is called q-optimal if $\mathbb{Q}^{(q)}$ minimizes $L^q(\mathbb{P})$-norm, i.e.*

$$\mathbb{E}\left[\left|\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}}\right|^q\right] = \inf_{\mathbb{Q} \in \mathscr{M}_q^s(\mathbb{P})} \mathbb{E}\left[\left|\frac{d\mathbb{Q}}{d\mathbb{P}}\right|^q\right].$$

In the context of stochastic volatility models driven by two correlated Brownian motions, the q-optimal martingale measure was studied in Hobson (2004) through the fundamental representation equation, which provides characterisation of the q-optimal martingale measure with respect to the real world measure. More recently, several authors studied this problem in the models with jumps. Jeanblanc, Klöppel and Miyahara (2007) provide sufficient conditions to identify the q-optimal martingale measure for exponential Lévy processes, however the authors do not consider signed martingale measures, only equivalent ones. For this reduced optimisation problem, they show that it suffices to consider equivalent martingale measures that preserve the Lévy property and this reduces the problem to a deterministic convex optimization problem. In Bender and Niethammer (2007), the q-optimal signed martingale measure is identified in the exponential Lévy models, and the proof is based on a hedging argument and duality for convex optimisation. The authors restrict the values of q to $q = \frac{2m}{2m-1}$ for some $m \in \mathbb{N}$, and they also study the convergence of the q-optimal signed martingale measures to the minimal entropy martingale measure. In the general semimartingale model, under some assumptions, the q-optimal martingale measure and q-optimal hedging problem was studied in Arai (2006). Also, in Sabanis (2008), necessary and sufficient conditions for the candidate measure to be q-optimal are presented.

For the remainder of the section, we assume that Assumption 3.2 holds.

Similarly to the previous section, the process of finding the optimal martingale measure takes two steps: finding the measure that takes certain form, and a verification procedure that guarantees the optimality. Let K_p^s denote the space spanned by the elementary stochastic integrals of the form $h(S_{T_2} - S_{T_1})$ where h is bounded, $\mathcal{F}_{T_1}$-measurable random variable and $T_1 \leq T_2$ are stopping times such that the stopped process S_{T_2} is bounded. Let K_p denote the closure of space K_p^s in $L^p(\mathbb{P})$. The following proposition gives a characterization of the q-optimal martingale measure.

Proposition 3.14. *[Sabanis (2008), Theorem 2.4]*
Assume that $\mathscr{M}_q^s(\mathbb{P}) \neq \varnothing$. If $\mathbb{Q}^{(q)}$ is the q-optimal signed martingale measure, then

$$\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}} = C_p \operatorname{sgn}\left(1 - \frac{f_p}{p-1}\right)\left|1 - \frac{f_p}{p-1}\right|^{\frac{1}{q-1}} \tag{3.30}$$

for some $f_p \in K_p$ and some constant C_p.

There exists an alternative characterization of the q-optimal signed martingale measure, which mirrors Definition 3.4 and Proposition 3.5, cf. Arai (2006). But first, we need to extend the Definition 3.7 of the space of admissible strategies.

Definition 3.15. *A strategy $\vartheta \in L(S)$ is admissible, if there exists a sequence $(\vartheta^{(n)})_{n\in\mathbb{N}}$ of simple strategies such that*

$$\int_0^T \vartheta dS = L^p - \lim_{n\to\infty} \int_0^T \vartheta^n dS, \int_0^t \vartheta dS = \lim_{n\to\infty} \int_0^t \vartheta^n dS \quad \text{in probability for all } t.$$

The space of admissible strategies is denoted by Θ^p.

It is proved in Theorem 2.1 of Xia and Yan (2006) that $G_T(\Theta^p)$ is closed in $L^p(\mathbb{P})$, i.e. $G_T(\Theta^p) = K_p$, under the following two assumptions assumptions, which we adopt as well:

Assumption 3.3. $\mathscr{M}_q^e \neq \varnothing$.

Assumption 3.4. *The process S is locally in $L^p(\mathbb{P})$ in the sense that S is $\mathcal{F}$-adapted, right continuous with left limits, and there exists a sequence $(\tau_n)_{n\geq 1}$ of localizing stopping times increasing to T such that for any $n \geq 1$, $\{S_\tau, \tau \text{ stopping time}, \tau \leq \tau_n\} \in L^p(\mathbb{P})$.*

Remark 3.3. By Xia and Yan (2006), Remark 2.3, under Assumptions 3.3 and 3.4, for any $f \in K_p$ there exists a predictable S-integrable process ϑ such that $f = \int_0^T \vartheta dS$ and $\int \vartheta dS$ is uniformly integrable under each $\mathbb{Q} \in \mathscr{M}_q^e$.

Now going back to alternative characterisation of the q-optimal signed martingale measure, c.f. Arai (2006), using our space of admissible strategies, we have the following:

Definition 3.16. *A process $\beta \in L(S)$ is called an adjustment process in $L^p(\mathbb{P})$-sense, if the following two conditions hold:*

1. $\beta \mathcal{E}\left(-\int \beta dS\right)_- \in \Theta^p$,
2. $\mathbb{E}\left[\operatorname{sgn}\left(\mathcal{E}\left(-\int \beta dS\right)_T\right) |\mathcal{E}\left(-\int \beta dS\right)_T|^{p-1} G_T(\vartheta)\right] = 0$ *for all* $\vartheta \in \Theta^p$.

The following Theorem and Proposition are due to Arai (2006). We provide full proofs, as Arai (2006) uses different space of admissible strategies.

Theorem 3.17. *[Arai (2006), Theorem 3.1] The signed measure $\mathbb{Q}^{(q)} \in \mathscr{M}_q^s$ is q-optimal signed martingale measure if and only if for all $\mathbb{Q} \in \mathscr{M}_q^s$*

$$\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \operatorname{sgn}\left(\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}}\right) \left|\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}}\right|^{q-1}\right] = const. \tag{3.31}$$

Proof. Let $\mathbb{Q}^{(q)}$ be a signed martingale measure such that for all $\mathbb{Q} \in \mathscr{M}_q^s$ equality (3.31) holds. Then we have

$$\begin{aligned}
\mathbb{E}\left[\left|\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}}\right|^q\right] &= \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \operatorname{sgn}\left(\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}}\right) \left|\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}}\right|^{q-1}\right] \\
&\leq \frac{1}{p}\mathbb{E}\left[\left|\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}}\right|^{p(q-1)}\right] + \frac{1}{q}\mathbb{E}\left[\left|\frac{d\mathbb{Q}}{d\mathbb{P}}\right|^q\right] \\
&= \frac{1}{p}\mathbb{E}\left[\left|\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}}\right|^q\right] + \frac{1}{q}\mathbb{E}\left[\left|\frac{d\mathbb{Q}}{d\mathbb{P}}\right|^q\right],
\end{aligned}$$

which implies

$$\mathbb{E}\left[\left|\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}}\right|^q\right] \leq \mathbb{E}\left[\left|\frac{d\mathbb{Q}}{d\mathbb{P}}\right|^q\right].$$

This shows that $\mathbb{Q}^{(q)}$ is q-optimal signed martingale measure.

On the other hand, assume that $\mathbb{Q}^{(q)}$ is q-optimal signed martingale measure. By Proposition 3.14 there exists $\theta \in \Theta^p$, such that $d\mathbb{Q}/d\mathbb{P} = C_p \operatorname{sgn}(1 - G_T(\theta))|1 - G_T(\theta)|^{p-1}$. Thus, for any $\mathbb{Q} \in \mathscr{M}_q^s$, we have

$$\begin{aligned}\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \operatorname{sgn}\left(\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}}\right)\left|\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}}\right|^{q-1}\right] &= \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \operatorname{sgn}(C_p(1 - G_T(\theta)))\left|C_p|1 - G_T(\theta)|^{(p-1)}\right|^{(q-1)}\right] \\ &= \operatorname{sgn}(C_p)|C_p|^{q-1}\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}(1 - G_T(\theta))\right] \\ &= \operatorname{sgn}(C_p)|C_p|^{q-1}.\end{aligned}$$

□

Proposition 3.18. *[Arai (2006), Theorem 3.4] Assume that $\mathscr{M}_q^s(\mathbb{P}) \neq \varnothing$. If β is an adjustment process in $L^p(\mathbb{P})$-sense, then there exists a signed martingale measure $\mathbb{Q}^{(q)} \in \mathscr{M}_q^s$ such that*

$$\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}} = \frac{\operatorname{sgn}\left(\mathcal{E}\left(-\int \beta dS\right)_T\right)|\mathcal{E}\left(-\int \beta dS\right)_T|^{p-1}}{\mathbb{E}\left[\operatorname{sgn}\left(\mathcal{E}\left(-\int \beta dS\right)_T\right)|\mathcal{E}\left(-\int \beta dS\right)_T|^{p-1}\right]} \tag{3.32}$$

and $\mathbb{Q}^{(q)}$ is the q-optimal signed martingale measure.

Proof. Let $\bar{Z}$ be the solution to $\bar{Z} = 1 - \int \beta \bar{Z}_- dS$, and let $\dot{Z} = \operatorname{sgn}(\bar{Z})|\bar{Z}|^{p-1}$. For any $\mathbb{Q} \in \mathscr{M}_q^s$, using the properties of the adjustment process, we have

$$\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \operatorname{sgn}(\dot{Z}_T)|\dot{Z}_T|^{q-1}\right] = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\bar{Z}_T\right] = 1.$$

Due to assumption $\mathscr{M}_q^s(\mathbb{P}) \neq \varnothing$, $\bar{Z}_T$ cannot be $\mathbb{P}$-a.s. equal to zero. Using that and the second property of the adjustment process,

$$\mathbb{E}\left[\dot{Z}_T\right] = \mathbb{E}\left[\dot{Z}_T\bar{Z}_T\right] = \mathbb{E}\left[|\bar{Z}_T|^p\right] > 0,$$

which shows that $\mathbb{Q}^{(q)}$ is well defined by (3.32) and q-optimal by Theorem (3.17). □

Looking at the form of equation (3.32), it is clear that the key to the first step, that is finding the candidate martingale measure, is to find the expression of r-th root of the absolute value of stochastic exponential. Define the function

$$g_r(y) := \operatorname{sgn}(y + 1)|y + 1|^r - 1. \tag{3.33}$$

Then for $r > 0$, the following identity

$$\mathcal{E}\left(\int H_u dB_u + \int\int h(u,x)N(du,dx)\right) =$$
$$\operatorname{sgn}\left(\mathcal{E}\left(r\int H_u dB_u + \int\int g_r(h(u,x))N(du,dx)\right)\right)$$
$$\times\left|\mathcal{E}\left(r\int H_u dB_u + \int\int g_r(h(u,x))N(du,dx)\right)\right|^{\frac{1}{r}}$$
$$\times\exp\left(\frac{r-1}{2}\int H_u^2 du + \frac{1}{r}\int\int \big(g_r(h(u,x)) - rh(u,x)\big)\nu(du,dx)\right) \tag{3.34}$$

holds directly from definition of g_r by noting that

$$\operatorname{sgn}\left(\mathcal{E}\left(\int H_u dB_u + \int\int h(u,x)N(du,dx)\right)\right) =$$
$$\operatorname{sgn}\left(\mathcal{E}\left(r\int H_u dB_u + \int\int g_r(h(u,x))N(du,dx)\right)\right)$$

and

$$\left|\mathcal{E}\left(\int H_u dB_u + \int\int h(u,x)N(du,dx)\right)\right|^r =$$
$$\left|\mathcal{E}\left(r\int H_u dB_u + \int\int g_r(h(u,x))N(du,dx)\right)\right|$$
$$\times\exp\left(\frac{r(r-1)}{2}\int H_u^2 du + \int\int \big(g_r(h(u,x)) - rh(u,x)\big)\nu(du,dx)\right).$$

We can now deduce the form of the martingale condition from the equation (3.34) by looking at how to express the integral of the form

$$r\int H_u dB_u + \int\int g_r(h(u,x))N(du,dx)$$

as an integral with respect to the price process S. Define the following mapping (dependence on V_{t-} omitted from notation)

$$\Phi(q,\lambda_t,t) = -\lambda_t S_{t-}\sigma_t^2 + b_t + \delta_t\mathbb{E}(J_1) + \delta_t\int_{\mathbb{R}} x g_{\frac{1}{q-1}}\big(-(q-1)\delta_t\lambda_t S_{t-}x\big)\nu(dx). \tag{3.35}$$

The martingale condition is given by the equation $\Phi(q,\lambda_t,t) = 0$ for all $t\in[0,T]$. The following Theorem provides sufficient condition for such stochastic process $(\lambda_t)_{0\le t\le T}$ to exist.

Theorem 3.19. *Assume that the Lévy process J has a finite p-th absolute moment. Then $\Phi(q,\lambda_t,t) < \infty$ and there exists a unique predictable process $(\lambda_t)_{0\le t\le T}$ such that $\Phi(q,\lambda_t,t) = 0$, almost surely.*

Proof. Fix q and t. First we show that the integral in the definition of $\Phi(q, \lambda_t, t)$ is finite for any $\lambda_t \in \mathbb{R}$. By Taylor expansion around zero

$$g_{\frac{1}{q-1}}(y) = (p-1)y + O(y^2).$$

Thus for any $\lambda_t \in \mathbb{R}$, we have

$$|xg_{\frac{1}{q-1}}(-(q-1)\delta_t S_{t-}\lambda_t x)| = |\delta_t S_{t-}\lambda_t|x^2 + O(|x|^3)$$

which is ν-integrable over $(-1, 1)$ by the definition of the Lévy measure. On the other hand, for any $\lambda_t \in \mathbb{R}$ and $|x| > 1$, we have

$$\begin{aligned} |xg_{\frac{1}{q-1}}(-(q-1)\delta_t S_{t-}\lambda_t x)| &\leq (|-\delta_t S_{t-}\lambda_t x + 1|^{(p-1)} + 1)|x| \\ &\leq ((|\delta_t S_{t-}\lambda_t x| + 1)^{(p-1)} + 1)|x| \\ &\leq (|\delta_t S_{t-}\lambda_t| + 1)^{(p-1)}|x|^p + |x|. \end{aligned} \tag{3.36}$$

The right-hand side of (3.36) is ν-integrable for $|x| \geq 1$ by assumption of the finite p-th absolute moment. Thus the integral is finite for any $\lambda_t \in \mathbb{R}$. The existence and uniqueness of $(\lambda_t)_{0\leq t\leq T}$ then follows from the fact that $xg_{\frac{1}{q-1}}(-(q-1)\delta_t S_{t-}\lambda_t x)$ is continuous and monotonically decreasing in λ with

$$\lim_{\lambda\downarrow-\infty} \Phi(q, \lambda_t, t) = \infty \quad \text{and} \quad \lim_{\lambda\uparrow\infty} \Phi(q, \lambda_t, t) = -\infty.$$

The predictability of the random process λ follows from the predictability of the remaining quantities appearing in the expression of $\Phi(q, \lambda_t, \cdot)$. $\square$

Proposition 3.20. *Assume there exists a solution to* $\Phi(q, \lambda_t, t) = 0$ *and that* $b_t + \delta_t \mathbb{E}(J_1) > 0$. *Then* $\lambda_t \in \left(0, \dfrac{b_t + \delta_t \mathbb{E}(J_1)}{S_{t-}\sigma_t^2}\right)$.

Proof. The proof is identical to the second part of proof of Proposition 3.11 after replacing Ψ by Φ. $\square$

Similarly to Section 3.3, we also introduce alternative space of admissible strategies. For any special semimartingale Y with Doob-Meyer decomposition $Y = Y_0 + M^Y + A^Y$, with $M_0^Y = A_0^Y = 0$, define

$$\| Y \|_{\mathscr{H}^p} = \| ([M^Y, M^Y]_T)^{\frac{1}{2}} + \mathrm{Var}(A^Y)_T \|_{L^p(\mathbb{P})} .$$

Then Y belongs to the space $\mathscr{H}^p$ if $\| Y \|_{\mathscr{H}^p} < \infty$. We denote by Θ^{H_p} the set of following strategies

$$\Theta^{H_p} := \{\vartheta \in L(S) : \int \vartheta dS \in \mathscr{H}^p\}.$$

Remark 3.4. Under assumption of Proposition 3.20, the sufficient condition for the solution λ to $\Phi(q, \lambda, t) = 0$ to be in Θ^{H_p} is given by

$$\mathbb{E}\left[\left(\left(\int_0^T (b_t + \delta_t a)^2 (dt + \frac{\delta_t^2}{\sigma_t^2}\int_{\mathbb{R}} x^2 dJ)\right)^{\frac{1}{2}} + \int_0^T \frac{(b_t + \delta_t a)^2}{\sigma_t^2} dt\right)^p\right] < \infty.$$

This follows directly from definition of Θ^{H_p} and Proposition 3.20.

Theorem 3.21 is analogous to the first statement of Corollary 2.9 in Černý and Kallsen (2007) for $p = 2$.

Theorem 3.21. $\Theta^{H_p} \subset \Theta^p$.

Proof. For $\vartheta \in \Theta^{H_p}$ we have $\mathbb{E}[\sup_{t\in[0,T]} |\int_0^t \vartheta dS|^p] < \infty$ by Protter (2004), Theorem V.2, and thus by Xia and Yan (2006), Lemma 2.2, $\int_0^T \vartheta dS \in K_p$. Let $\int_0^T \theta^{(n)} dS$ be an approximating sequence to $\int_0^T \vartheta dS$. Theorem 1.2 in Delbaen and Schachermayer (1996a) shows that $\int_0^T \vartheta dS$ can be written as $\int_0^T \theta dS$ for some $\theta \in L(S)$ such that $\int_0^t \theta^{(n)} dS$ converges in probability to $\int_0^t \theta dS$ for each $t \in [0, T]$. Therefore $\theta \in \Theta^p$. It remains to show that $\theta = \vartheta$. Let $\mathbb{Q} \in \mathscr{M}_q^e$, which by Assumption 3.3 is nonempty, and denote its density process by $Z^{\mathbb{Q}}$. Since $\sup_{t\in[0,T]} |Z^{\mathbb{Q}}|^q \in L^q(\mathbb{P})$ and both $\int \vartheta dS$ and $\int \theta dS$ are L^p-semimartingales, in the sense of Assumption 3.4, cf. Remark 3.3, we have by Jacod and Shiryaev (2002), Proposition 1.47 (c), that both $Z^{\mathbb{Q}} \int \vartheta dS$ and $Z^{\mathbb{Q}} \int \theta dS$ are true martingales and thus $\theta = \vartheta$. □

Define the processes

$$\dot{H} \triangleq -\sigma S_{-}\lambda,$$
$$\dot{h}(\cdot, x) \triangleq g_{\frac{1}{q-1}}\big(-(q-1)\delta S_{-}\lambda x\big),$$

where λ satisfies $\Phi(q, \lambda_t, t) = 0$ for all $t \in [0, T]$.

Theorem 3.22. *Assume that the stochastic process λ satisfies $\Phi(q, \lambda_t, t) = 0$ for all $t \in [0, T]$, that $\mathcal{E}\left(-\int (q-1)\lambda dS\right)_T \in L^p(\mathbb{P})$ and that the random variable*

$$\exp\left(\left(1 - \frac{q}{2}\right)\int_0^T \dot{H}_u^2 du - \int_0^T \lambda_u dA_u + \int_0^T\int_{\mathbb{R}} (\delta_u S_{u-}\lambda_u x + \dot{h}(u, x))\nu(du, dx)\right)$$

is integrable and has positive expectation. Further assume that $\iint_{\mathbb{R}} |\dot{h}(s,x)| J(ds,dx)$ *is locally* $\mathbb{P}$*-integrable, and that both*

$$\mathcal{E}\left(\int \dot{H}_u dB_u + \iint_{\mathbb{R}} \dot{h}(u,x) N(du,dx) + \int F_s dB_u^V + \iint_{\mathbb{R}} f(u,x) N^V(ds,dx)\right)$$

and

$$\mathcal{E}\left(-\int (q-1)\lambda dM\right)$$

are true martingales, where F *and* f *satisfy*

$$\begin{aligned} C_p = {} & \mathcal{E}\left(\int F_u dB_u^V + \int\int f(u,x) N^V(du,dx)\right)_T \exp\left(\frac{q}{2}\int_0^T (\sigma_u S_{u-}\lambda_u)^2 du\right) \\ & \times \exp\left(\int_0^T \int (-\delta_u S_{u-}\lambda_u x + g_{\frac{1}{q-1}}(-(q-1)\delta_u S_{u-}\lambda_u x)^2)\nu(du,dx)\right) \\ & \times \exp\left(-\int_0^T \int g_{\frac{1}{q-1}}(-(q-1)\delta_u S_{u-}\lambda_u x)\nu(du,dx)\right). \end{aligned} \tag{3.37}$$

The density of the q-optimal signed martingale measure with respect to $\mathbb{P}$ *is given by equation* (3.32) *with adjustment process* $\beta = (q-1)\lambda$.

Proof. First we show that by choosing $\dot{H}_u = -\sigma_u S_{u-}\lambda_u$ and $\dot{h}(u,x) = g_{\frac{1}{q-1}}\big(-(q-1)\delta_u S_{u-}\lambda_u x\big)$, the density of the signed martingale measure will take the form (3.32) when $\beta = (q-1)\lambda$, and C_p in Proposition 3.14 is given by (3.37). We have

$$\begin{aligned} & \mathcal{E}\left(\int \dot{H}_u dB_u + \int\int \dot{h}(u,x) N(du,dx)\right)_T \\ = \quad & \mathcal{E}\left(\int \dot{H}_u d\dot{B}_u + \int\int \dot{h}(u,x) \dot{N}(du,dx)\right)_T \qquad (3.38) \\ & \times \exp\left(\int_0^T (\sigma_u S_{u-}\lambda_u)^2 du + \int_0^T \int g_{\frac{1}{q-1}}(-(q-1)\delta_u S_{u-}\lambda_u x))^2 \nu(du,dx)\right). \end{aligned}$$

Now apply identity (3.34) to expression on line (3.38) with $r = q-1$ and note that for any $y \in \mathbb{R}$, $g_r(g_{\frac{1}{r}}(y)) = y$. This yields

$$\begin{aligned} & \mathcal{E}\left(\int \dot{H}_u dB_u + \int\int \dot{h}(u,x) N(du,dx) + \int F_u dB_u^V + \int\int f(u,x) N^V(du,dx)\right)_T \\ & = C_p \operatorname{sgn}\left(\mathcal{E}\left(-\int (q-1)\lambda_u dS_u\right)_T\right) \left|\mathcal{E}\left(-\int (q-1)\lambda_u dS_u\right)_T\right|^{\frac{1}{q-1}}. \end{aligned}$$

Using now definitions of the gain process the stochastic exponential, we recover equations of the form given in (3.30) and (3.32).

In light of Proposition 3.18, it remains to show that $(q-1)\lambda$ is an adjustment process in the $L^p(\mathbb{P})$-sense. We start by showing that the second property of the adjustment process holds. Because A is continuous and finite variation process, we have

$$\begin{aligned}
&\operatorname{sgn}\left(\mathcal{E}\left(-\textstyle\int(q-1)\lambda dS\right)_T\right)|\mathcal{E}\left(-\textstyle\int(q-1)\lambda dS\right)_T|^{p-1} \\
&\quad= \operatorname{sgn}\left(\mathcal{E}\left(-\textstyle\int(q-1)\lambda dM\right)_T\right)|\mathcal{E}\left(-\textstyle\int(q-1)\lambda dM\right)_T|^{p-1} \\
&\qquad\times \operatorname{sgn}\left(\exp(-\textstyle\int(q-1)\lambda dA)\right)|\exp(-\textstyle\int(q-1)\lambda dA)|^{p-1}.
\end{aligned}$$

Using identity (3.34) we get

$$\begin{aligned}
\operatorname{sgn}\left(\mathcal{E}\left(-\textstyle\int(q-1)\lambda dM\right)_T\right)|\mathcal{E}\left(-\textstyle\int(q-1)\lambda dM\right)_T|^{p-1} &= \mathcal{E}\left(\textstyle\int \dot{H}dB + \int\int \dot{h}N\right)_T \\
\times \exp\left(-\frac{q-2}{2}\int \dot{H}^2 du - \frac{1}{q-1}\int\int \left(g_{q-1}(\dot{h}) - (q-1)\dot{h}\right)\nu(du,dx)\right).
\end{aligned}$$

Now define a process D as

$$\begin{aligned}
D = \operatorname{sgn}&\left(\exp(-\int(q-1)\lambda dA)\right)\left|\exp\left(-\int(q-1)\lambda dA\right)\right|^{p-1} \\
&\times \exp\left(-\frac{q-2}{2}\int \dot{H}^2 du - \frac{1}{q-1}\int\int_{\mathbb{R}} \left(g_{q-1}(\dot{h}) - (q-1)\dot{h}\right)\nu(du,dx)\right),
\end{aligned}$$

and note that it is independent of S. This is because $\dot{H}$, $\dot{h}$ and λdA are all functions of the model parameters and functions of the product λS, which is independent of S and can be seen by looking at definition of λ, cf. equation (3.35). Therefore the martingale defined as $M_t^V = \mathbb{E}[D_T|\mathcal{F}_t]$ depends only on the volatility and the product

$$\mathcal{E}\left(\textstyle\int \dot{H}dB + \int\int \dot{h}N\right) M^V$$

is by the assumption of the theorem a true martingale measure, after scaling by $\mathbb{E}[D_T]$. Combining the results above, for any $\vartheta \in \Theta^p$ we have

$$\begin{aligned}
&\mathbb{E}\left[\operatorname{sgn}\left(\mathcal{E}\left(-\textstyle\int(q-1)\lambda dS\right)_T\right)|\mathcal{E}\left(-\textstyle\int(q-1)\lambda dS\right)_T|^{p-1}G_T(\vartheta)\right] \\
&\qquad= \mathbb{E}\left[\mathcal{E}\left(\textstyle\int \dot{H}dB + \int\int \dot{h}N\right)_T M_T^V G_T(\vartheta)\right] \\
&\qquad= 0,
\end{aligned}$$

which shows that second property of the adjustment process holds. The expectation above is finite due to assumption of the Theorem, i.e.

$$\mathbb{E}\left[|\left|\mathcal{E}\left(-\textstyle\int(q-1)\lambda dS\right)_T\right|^{p-1}G_T(\vartheta)|\right] \le \mathbb{E}\left[\left|\mathcal{E}\left(-\textstyle\int(q-1)\lambda dS\right)_T\right|^p\right]^{\frac{1}{q}}\mathbb{E}\left[|G_T(\vartheta)|^p\right]^{\frac{1}{p}} < \infty.$$

We now proceed to show that first property of the adjustment process holds as well. Let $\bar{Z}$ be the solution to $\bar{Z} = 1 - \int(q-1)\lambda\bar{Z}_- dS$, and let $\dot{Z} = \operatorname{sgn}(\bar{Z})|\bar{Z}|^{p-1}$. Further let $Z = 1 - \int(q-1)\lambda Z_- dM$ and let $K = \int(q-1)\lambda dA$. Note that similarly to $\hat{K}$, K is a continuous increasing process of finite variation. This can be seen by writing $K = \int(q-1)\lambda\hat{\lambda}d\langle M, M\rangle$, and noting that the product $\lambda_t\hat{\lambda}_t$ is always non-negative because both λ_t and $\hat{\lambda}_t$ have the same sign as $b_t + \delta_t\mathbb{E}(J_1)$ for all $t \in [0, T]$, c.f. equation (3.2) and Proposition 3.20. For $n \in \mathbb{N}$, define $T_n := \inf\{t > 0 : D_t \ge n\}$. In the first step we show that the integrand, $(q-1)\lambda 1_{[0,T_n]}\bar{Z}_-^{T_n}$, is admissible. We have

$$\begin{aligned}
\| \bar{Z}^{T_n} \|_{\mathscr{H}^p} &= \| Z^{T_n}\exp(-K^{T_n}) \|_{\mathscr{H}^p} \le \| Z^{T_n} \|_{\mathscr{H}^p} \\
&\le c_p\mathbb{E}[\big(\sup_{t\in[0,T]}|Z_t^{T_n}|\big)^p]^{\frac{1}{p}} && (3.39)\\
&\le c_p\frac{p}{p-1}\mathbb{E}[|Z_T^{T_n}|^p]^{\frac{1}{p}} && (3.40)\\
&\le c_p\frac{p}{p-1}\mathbb{E}[|Z_T^{T_n}|^p\exp(pK_T^{T_n})]^{\frac{1}{p}} \\
&= c_p\frac{p}{p-1}\mathbb{E}[|\bar{Z}_T^{T_n}|^p]^{\frac{1}{p}} \\
&= c_p\frac{p}{p-1}\mathbb{E}[\dot{Z}_T^{T_n}]^{\frac{1}{p}} = c_p\frac{p}{p-1}\mathbb{E}[\mathcal{E}\left(\textstyle\int \dot{H}dB + \int\int \dot{h}N\right)_T^{T_n} D_T^{T_n}]^{\frac{1}{p}} && (3.41)\\
&= c_p\frac{p}{p-1}n^{\frac{1}{p}}. && (3.42)
\end{aligned}$$

Line (3.39) follows from the Burkholder-Davis-Gundy inequality, cf Protter (2004), Theorem IV.48, where c_p is a constant independent of the local martingale Z. Line (3.40) follows from Doob's maximal inequality and assumption of the Theorem that Z is a true martingale. Line (3.41) is justified by using the second property of the adjustment process, and the fact that it holds also for the stopped process S_{T_n}, due to independence of T_n from S. Finally, the last equality (3.42) follows from the definition of the stopping time T_n. Thus the integrand $(q-1)\lambda 1_{(0,T_n]}\bar{Z}_-^{T_n} \in \Theta^{H_p}$ and therefore admissible by Theorem 3.21. Therefore the density of the q-optimal signed martingale measure for the stopped process S^{T_n} is given by $\dot{Z}_T^{T_n}/\mathbb{E}[\dot{Z}_T^{T_n}]$. This implies that $\dot{Z}_T^{T_n} \in L^q(\mathbb{P})$ for $n \in \mathbb{N}$, $\{\dot{Z}_T^{T_n}, n \in \mathbb{N}\}$ is uniformly integrable and therefore $\mathbb{E}[\dot{Z}_T^{T_n}] \to \mathbb{E}[\dot{Z}_T]$, cf. Gut (2005), Theorems 5.4.2 and 5.5.2. As $(q-1)\hat{\lambda}1_{(0,T_n]}$ is the adjustment process for the stopped process S^{T_n} for $n \in \mathbb{N}$, we have that

$$\mathbb{E}[\dot{Z}_T^{T_n}] = \mathbb{E}[\dot{Z}_T^{T_n}\bar{Z}_T^{T_n}] = \mathbb{E}[|\bar{Z}_T^{T_n}|^p],$$

and similarly, as the second property of the adjustment process was shown to hold in the first part of the proof, we have that $\mathbb{E}[\dot{Z}_T]=\mathbb{E}[|\bar{Z}_T|^p]$. Thus $\mathbb{E}[|\bar{Z}_T^{T_n}|^p] \to \mathbb{E}[|\bar{Z}_T|^p]$ which by Gut (2005), Theorem 5.5.2 implies that $\int_0^T \lambda 1_{(0,T_n]}\bar{Z}_-^{T_n}dS$ converges to $\int_0^T \lambda\bar{Z}_-dS$ in $L^p(\mathbb{P})$. This together with the admissibility of $(q-1)\lambda 1_{(0,T_n]}\bar{Z}_-^{T_n}$ now implies that there exists a sequence of simple trading strategies $(\vartheta^n)_{n\in\mathbb{N}}$ such that $\int_0^T \vartheta^n dS$ converges to $\int_0^T (q-1)\lambda\bar{Z}_-dS$ in $L^p(\mathbb{P})$. Note that all results in the proof so far also hold when T is replaced with any positive $t < T$. Thus we also have that $\int_0^t \vartheta^n dS$ converges to $\int_0^t (q-1)\lambda\bar{Z}_-dS$ in $L^p(\mathbb{P})$ and therefore also in probability. This shows that $(q-1)\lambda\bar{Z}_-$ is admissible.

This implies that $(q-1)\lambda$ is an adjustment process. From equation (3.37) we have that $\mathbb{E}[\dot{Z}_T] = \mathbb{E}[D_T]$, and by the assumption of the Theorem, $\mathbb{E}[\dot{Z}_T] > 0$. As $(q-1)\lambda$ is an adjustment process, $\mathbb{E}[\dot{Z}_T] = \mathbb{E}[|\bar{Z}_T|^p]$. Thus

$$\mathbb{E}\left[\left|\frac{\dot{Z}_T}{\mathbb{E}[\dot{Z}_T]}\right|^q\right] = \frac{\mathbb{E}[|\bar{Z}_T|^p]}{\mathbb{E}[\dot{Z}_T]^q} = \frac{1}{\mathbb{E}[\dot{Z}_T]^{q-1}} = \frac{1}{\mathbb{E}[D_T]^{q-1}} < \infty,$$

and therefore $\dot{Z}_T/\mathbb{E}[\dot{Z}_T]$ is the density of the q-optimal signed martingale measure.

□

Remark 3.5. Note that when $q = 2m$ or $q = \frac{2m}{2m-1}$ for some $m \in \mathbb{N}$, we can define $g_q(y)$ as $g_q(y) := (y+1)^q - 1$, and the form of q-optimal martingale measure is simplified to

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = C_p(1 - G_T(\theta))^{\frac{1}{q-1}}.$$

This is because in that case both $g_{q-1}(y)$ and $(1 - G_T(\theta))^{\frac{1}{q-1}}$ define real valued functions even though $y + 1$ or $1 - G_T(\theta)$ are negative.

In some applications, it is necessary to know the dynamics of the volatility under the martingale measure, i.e. one needs to find F and f appearing in the representation equation. By the assumption that model parameters depend only on the volatility, the process R defined as

$$R_t = \exp\left(\frac{q}{2}\int_0^t (\sigma_u S_{u-}\lambda_u)^2 du\right)$$
$$\exp\left(\int_0^t \int (-\delta_u S_{u-}\lambda_u x + g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u x)^2 - g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u x))\nu(du,dx)\right)$$

is a function of V_t only. Thus, one can recover F and f by using the martingale representation theorem, from which it follows that

$$\mathcal{E}\left(\int F_u dB_u^V + \int\int f(u,x)N^V(du,dx)\right)_t = \frac{\mathbb{E}[R_T|\mathcal{F}_t]}{\mathbb{E}[R_T]}.$$

When the q-optimal martingale measure is equivalent to $\mathbb{P}$, then

$$\mathbb{E}\left[\left|\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}}\right|^q\right] = \mathbb{E}\Bigg[\mathcal{E}\left(\int q\sigma_u S_{u-}\lambda dB_u + \int\int g_q((1-q)\delta_u S_{u-}\lambda_u x)N(du,dx)\right)_T$$
$$\mathcal{E}\left(\int F_u dB_u^V + \int\int f(u,x)N^V(du,dx)\right)_T C_p^{q-1}\Bigg]$$
$$= C_p^{q-1}.$$

This extends the result of Hobson (2004) for continuous stochastic volatility models, and in fact, this can be seen to hold for any semimartingale model as long as the q-optimal martingale measure is equivalent to $\mathbb{P}$.

We now provide a simple example in which it is possible in (3.35) to explicitly solve for λ that satisfies the martingale condition.

Example 3.3. *Let q=1.5 and let X be a Poisson process with rate 1. A Lévy process with bounded jumps has absolute moments of all orders, and 1.5-th moment in particular. By Theorem 3.19, there exists unique solution to $\Phi(1.5,\lambda_t,t)=0$. It is given by*

$$\lambda_t = \begin{cases} 2\dfrac{-(\sigma_t^2+\delta_t^2)+\sqrt{(\sigma_t^2+\delta_t^2)^2-\delta_t^3(b_t+\delta_t)}}{S_{t-}\delta_t^3}, & 2\sigma_t^2 - b_t\delta_t > 0; \\ \dfrac{-2b_t}{S_{t-}(2\sigma_t^2+\delta_t^2)}, & 2\sigma_t^2 - b_t\delta_t = 0; \\ 2\dfrac{\sigma_t^2-\delta_t^2-\sqrt{(\sigma_t^2-\delta_t^2)^2-\delta_t^3(b_t-\delta_t)}}{S_{t-}\delta_t^3}, & 2\sigma_t^2 - b_t\delta_t < 0, \end{cases}$$

Assuming that $\lambda \in \Theta^{H_3}$, the 1.5-optimal signed martingale measure is given by

$$\frac{d\mathbb{Q}^{(1.5)}}{d\mathbb{P}} = \frac{\operatorname{sgn}\left(\mathcal{E}\left(-\frac{1}{2}\int\lambda_u dS_u\right)_T\right)\left|\mathcal{E}\left(-\frac{1}{2}\int\lambda_u dS_u\right)_T\right|^2}{\mathbb{E}\left[\operatorname{sgn}\left(\mathcal{E}\left(-\frac{1}{2}\int\lambda_u dS_u\right)_T\right)\left|\mathcal{E}\left(-\frac{1}{2}\int\lambda_u dS_u\right)_T\right|^2\right]}.$$

3.5.1 Convergence to the minimal entropy martingale measure

In this section we study convergence of the q-optimal martingale measure to the minimal entropy martingale measure as $q \downarrow 1$. This problem was first studied by Grandits and Rheinländer (2002) in the continuous semimartingale setting. Considering only equivalent martingale measures, Jeanblanc, Klöppel and Miyahara (2007) study the convergence in the exponential Lévy models. Their results were generalized by Bender and Niethammer (2007), taking into account the fact that q-optimal martingale measure is signed measure in general. We will follow their work and extend the results to our stochastic volatility model driven by Lévy processes. Denote by $\lambda^{(q)}$ and $\lambda^{(e)}$ processes satisfying $\Phi(q, \lambda_t^{(q)}, t) = 0$ and $\Psi(\lambda_t^{(e)}, t) = 0$ for each $t \in [0, T]$, respectively. The following assumptions will be needed on the stochastic process λ that enters the definition of Φ and Ψ.

Assumption 3.5. *If $\delta_t < 0$ then*

$$\int_{x>1} e^{-0.28\lambda_t S_{t-}\delta_t x} \nu(dx) < \infty,$$

or if $\delta_t > 0$ then

$$\int_{x<-1} e^{0.28\lambda_t S_{t-}\delta_t x} \nu(dx) < \infty.$$

Assumption 3.6. *If $\delta_t \leq 0$ ($\delta_t \geq 0$) for all $t \in [0, T]$, then $\lambda S_{-}\delta$ is uniformly bounded from below (above) on $[0, T]$. Otherwise, $\lambda S_{-}\delta$ is uniformly bounded on $[0, T]$.*

Recall the definition of mapping $\Psi(\lambda, \cdot)$ given in (3.26). Under the assumption $b_t + \delta_t \mathbb{E}(J_1) > 0$, it was shown in Proposition 3.11 and Proposition 3.20 that both $\lambda_t^{(e)}$ and $\lambda_t^{(q)}$ are necessarily positive. Therefore we state the following Lemma only for the predictable process $\lambda > 0$.

Lemma 3.23. *Suppose the drift $b_t + \delta_t \mathbb{E}(J_1)$ is positive, the predictable process λ is positive, that $\int_{\mathbb{R}} x(e^{-\lambda_t S_{t-}\delta_t x} - 1)\nu(dx) < \infty$ for all t and that the Assumption 3.6 holds.*

(1) If $\delta_t < 0$ and the jumps of J are bounded from below or $\delta_t > 0$ and the jumps of

J are bounded from above, then

$$\lim_{q\downarrow 1}\Phi(q,\lambda_t,t)=\Psi(\lambda_t,t).$$

(2) If the Assumption 3.5 holds, then

$$\lim_{q\downarrow 1}\Phi(q,\lambda_t,t)=\Psi(\lambda_t,t).$$

Proof. (1a) Assume that $\delta_t>0$ and that jumps of J are bounded from above by some constant K. Then there exists q_0 such that for $x\le K$ and $q<q_0$, $(1-q)\delta_t S_{t-}\lambda_t x+1>0$. Noting that $g_{\frac{1}{q-1}}((1-q)\delta_t S_{t-}\lambda_t x)$ is monotonically decreasing in q for $q<q_0$, by monotone convergence, we get

$$\lim_{q\downarrow 1}\int_{x\le K} x g_{\frac{1}{q-1}}((1-q)\delta_t S_{t-}\lambda_t x)\nu(dx)=\int_{x\le K} x(e^{-\lambda_t S_{t-}\delta_t x}-1)\nu(dx).$$

By assumption of the Lemma, the limit on the right side is finite and thus

$$\lim_{q\downarrow 1}\Phi(q,\lambda_t,t)=\Psi(\lambda_t,t).$$

(1b) If we now assume that $\delta_t<0$ and that jumps of J are bounded from below by some constant K, the proof is similar to (1a).
(2a) Assume that $\delta_t>0$. In view of (1a), we need to prove the following

$$\lim_{q\downarrow 1}\int_{x>1} x g_{\frac{1}{q-1}}((1-q)\delta_t S_{t-}\lambda_t x)\nu(dx)=\int_{x>1} x(e^{-\lambda_t S_{t-}\delta_t x}-1)\nu(dx).$$

Note that on the set $\{x>1;(1-q)\delta_t S_{t-}\lambda_t x\ge -2\}$

$$|g_{\frac{1}{q-1}}((1-q)\delta_t S_{t-}\lambda_t x)|\le 2$$

and on the set $\{x>1;(1-q)\delta_t S_{t-}\lambda_t x<-2\}$

$$\begin{aligned}|g_{\frac{1}{q-1}}((1-q)\delta_t S_{t-}\lambda_t x)| &= |(1-q)\delta_t S_{t-}\lambda_t x+1|^{\frac{1}{q-1}}+1\\ &= ((q-1)\delta_t S_{t-}\lambda_t x-1)^{\frac{1}{q-1}}+1\\ &\le e^{0.28\lambda_t S_{t-}\delta_t x}+1,\end{aligned}$$

because $y\le e^{0.28y}+1$ for all $y\in\mathbb{R}$. Thus we get the claim by the dominated convergence theorem.

(2b) When $\delta_t < 0$, the similar reasoning as in (2a) yields

$$\lim_{q\downarrow 1} \int_{x<-1} x g_{\frac{1}{q-1}}((1-q)\delta_t S_{t-}\lambda_t x)\nu(dx) = \int_{x<-1} x(e^{-\lambda_t S_{t-}\delta_t x} - 1)\nu(dx).$$

□

Remark 3.6. Recall that in order for S to remain positive, the jumps of J need to be bounded from below when $\delta(t, V_{t-}) > 0$ for some t and bounded from above, when $\delta(t, V_{t-}) < 0$ for some t. Thus, the assumption in Lemma 3.23 (1) implies that J needs to have bounded jumps.

We now prove almost sure convergence of predictable processes $\lambda^{(q)}$ to a predictable process $\lambda^{(e)}$ for $q \downarrow 1$.

Theorem 3.24. *Assume the drift $b_t + \delta_t \mathbb{E}(J_1)$ is positive and there exists κ such that the Assumption 3.6 holds for the process $\lambda \triangleq \lambda^{(e)} + \kappa$. If there exists $\varepsilon > 0$ such that*

$$\int_0^T \int_{|x|>1} e^{-(\lambda_t^{(e)}+\varepsilon)\delta_t S_{t-}x} \nu(dx)dt < \infty \tag{3.43}$$

or the Assumption 3.5 holds for the process $\lambda \triangleq \lambda^{(e)} + \varepsilon$, then $\lambda^{(q)} \to \lambda^{(e)}$ almost surely for a fixed t as $q \downarrow 1$, and uniformly in t, for $t \in [0, T]$.

Proof. Fix t. We first consider the case when either $\delta_t < 0$ and the jumps of J are bounded from below or $\delta_t > 0$ and the jumps of J are bounded from above. Due to assumption (3.43), $\Psi(\lambda_t^{(e)} + \epsilon, t)$ and $\Psi(\lambda_t^{(e)} - \epsilon, t)$ are well-defined for $0 < \epsilon < \varepsilon$, and as Ψ monotonically decreasing

$$\Psi(\lambda_t^{(e)} - \epsilon, t) > 0 = \Psi(\lambda_t^{(e)}, t) > \Psi(\lambda_t^{(e)} + \epsilon, t).$$

For all such $\epsilon < \kappa$ and sufficiently small q, due to $\delta_t < 0$ and the jumps of J are bounded from below or $\delta_t > 0$ and the jumps of J are bounded from above, the expression $(1-q)\delta_t S_{t-}\lambda_t x + 1$ is positive for any $\lambda_t \in (\lambda_t^{(e)} - \epsilon, \lambda_t^{(e)} + \epsilon)$ and x. By Lemma 3.23 (1), there exists $q(\epsilon) > 1$ such that for $q \in (1, q(\epsilon))$

$$\Phi(q, \lambda_t^{(e)} - \epsilon, t) > 0 > \Phi(q, \lambda_t^{(e)} + \epsilon, t).$$

The claim follows from the continuity and monotonicity of Φ which implies that $\lambda_t^{(q)}$ is unique and satisfies $\lambda_t^{(q)} \in (\lambda_t^{(e)} - \epsilon, \lambda_t^{(e)} + \epsilon)$.

The proof in the case when $\delta_t < 0$ and the jumps of J are not bounded from below or $\delta_t > 0$ and the jumps of J are not bounded from above is similar to above, using Lemma 3.23 (2). □

Theorem 3.25. *Assume that the minimal entropy martingale measure exists and is given by*

$$\frac{d\mathbb{Q}^{(e)}}{d\mathbb{P}} = \frac{\exp(\int \lambda^{(e)} dS)}{\mathbb{E}[\exp(\int \lambda^{(e)} dS)]}.$$

Under the assumption of Theorem 3.24 and Theorem 3.22, the q-optimal martingale measure converges in probability to the minimal entropy martingale measure as $q \downarrow 1$.

Proof. By Yor's formula (cf. Protter (2004), Theorem II.38)

$$\begin{aligned}
&\mathcal{E}\left(-\int \sigma_u S_{u-}\lambda_u^{(q)} dB_u + \int\int g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)} x) N(du,dx)\right)_T \\
&= \mathcal{E}\left(-\int \sigma_u S_{u-}\lambda_u^{(q)} dB_u\right)_T \\
&\quad \times \mathcal{E}\left(\int\int_{|x|>1} g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)} x) N(du,dx)\right)_T \qquad (3.44)\\
&\quad \times \mathcal{E}\left(\int\int_{|x|\le 1} g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)} x) N(du,dx)\right)_T. \qquad (3.45)
\end{aligned}$$

By Theorem 3.24, for each $\epsilon \in (0, \min(\varepsilon, \kappa))$ there exists $q(\epsilon) > 1$ such that $|\lambda_t^{(q)} - \lambda_t^{(e)}| < \epsilon$ for all $q \in (1, q(\epsilon))$ and $t \in [0,T]$. Define the process $\lambda^d \triangleq \lambda^{(q(\epsilon))} + \lambda^{(e)}$. The random predictable process $\sigma S_- \lambda^d$ is integrable and $\sigma S_- \lambda^{(q)} < \sigma S_- \lambda^d$ for $q \in (1, q(\epsilon))$. By Theorem 3.24, the process $\sigma S_- \lambda^{(q)}$ converges almost surely to $\sigma S_- \lambda^{(e)}$ for $q \in (1, q(\epsilon)), q \downarrow 1$. Thus by dominated convergence for stochastic integrals, cf. Theorem 32, Chapter IV in Protter (2004), in probability

$$\int_0^T \sigma_u S_{u-}\lambda_u^{(q)} dB_u \to \int_0^T \sigma_u S_{u-}\lambda_u^{(e)} dB_u.$$

By continuous mapping theorem and simple solution to SDE governed by stochastic exponential for continuous martingales, we then have, in probability

$$\mathcal{E}\left(\int \sigma_u S_{u-}\lambda_u^{(q)} dB_u\right)_T \to \mathcal{E}\left(\int \sigma_u S_{u-}\lambda_u^{(e)} dB_u\right)_T. \qquad (3.46)$$

Using equation (2.13), the second term given by (3.44) can be written as

$$\mathcal{E}\left(\int\int_{|x|>1} g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)}x)N(du,dx)\right)_T =$$
$$\exp\left(-\int_0^T\int_{|x|>1} g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)}x)\nu(dx)du\right)$$
$$\times\prod_{u\le T} 1_{|\triangle X_u|>1}\big(g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)}\triangle J_u)+1\big). \tag{3.47}$$

By Theorem 3.24, $g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)}x)$ converges almost surely to $\exp(-\delta_u S_{u-}\lambda_u^{(e)}x)-1$ as $q\downarrow 1$. Note that $|g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)}x)|$ is dominated on $\{x,|x|>1\}$ by ν-integrable function given by assumption (3.43). On the other hand, the expression (3.47) has only finitely many factors. Thus, in probability

$$\mathcal{E}\left(\int\int_{|x|>1} g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)}x)N(du,dx)\right)_T \to$$
$$\mathcal{E}\left(\int\int_{|x|>1} \big(\exp(-\delta_u S_{u-}\lambda_u^{(e)}x)-1\big)N(du,dx)\right)_T. \tag{3.48}$$

Due to Assumption 3.6, there exist positive constants ϵ and $q>1$ such that for $|x|<1$ and for all $t\in[0,T]$, $g_{\frac{1}{q-1}}((1-q)\delta_t S_{t-}\lambda_t^{(q)}x)>\epsilon-1$. Thus, the last term given by (3.45) can be rewritten as

$$\mathcal{E}\left(\int\int_{|x|\le 1} g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)}x)N(du,dx)\right)_T =$$
$$\exp\left(-\int_0^T\int_{|x|\le 1}\big(g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)}x)-\log(g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)}x)+1)\big)\nu(dx)du\right)$$
$$\times\exp\left(\int_0^T\int_{|x|\le 1}\log\big(g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)}x)+1\big)N(du,dx)\right). \tag{3.49}$$

The integrand in the second line of equation (3.49) is dominated by ν-integrable function on $\{x,|x|\le 1\}$. By Taylor theorem

$$\log\big(g_{\frac{1}{q-1}}(y)+1\big)=g_{\frac{1}{q-1}}(y)-\frac{1}{2(\xi(y)+1)^2}g_{\frac{1}{q-1}}(y)^2 \tag{3.50}$$

for intermediate point $\xi(y)\in[0,g_{\frac{1}{q-1}}(y)]$, and for sufficiently small q

$$g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)}x)=((1-q)\delta_u S_{u-}\lambda_u^{(q)}\varphi(x)+1)^{\frac{1}{q-1}-1}\delta_u S_{u-}\lambda x\le Kx \tag{3.51}$$

for intermediate point $\varphi(x)\in[0,x]$ and some constant K independent of x. Thus

$$|g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)}x)-\log(g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)}x)+1)|\le\frac{g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)}x)^2}{2\epsilon^2}$$
$$\le\frac{K^2}{2\epsilon^2}x^2.$$

Thus by dominated convergence

$$\int_{|x|\le 1} \big(g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)} x) - \log(g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)} x) + 1)\big)\nu(dx) \to$$
$$\int_{|x|\le 1} \big(\exp(-\delta_u S_{u-}\lambda_u^{(e)} x) - 1 + \delta_u S_{u-}\lambda_u^{(e)} x\big)\nu(dx).$$

For the term on line (3.49), we have by the isometry formula

$$\mathbb{E}\left[\left|\int_0^T\int_{|x|\le 1} \big(\log\big(g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)} x) + 1\big) + \delta_u S_{u-}\lambda_u^{(e)} x\big) N(du,dx)\right|^2\right] =$$
$$\mathbb{E}\left[\int_0^T\int_{|x|\le 1} \big|\log\big(g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)} x) + 1\big) + \delta_u S_{u-}\lambda_u^{(e)} x\big|^2 \nu(dx)du\right]. \quad (3.52)$$

The integrand in the equation (3.52) is dominated by ν-integrable function on $\{x, |x| \le 1\}$ by (3.50) and (3.51). Thus in probability

$$\mathcal{E}\left(\int\int_{|x|\le 1} g_{\frac{1}{q-1}}((1-q)\delta_u S_{u-}\lambda_u^{(q)} x) N(du,dx)\right)_T \to$$
$$\mathcal{E}\left(\int\int_{|x|\le 1} \big(\exp(-\delta_u S_{u-}\lambda_u^{(e)} x) - 1\big) N(du,dx)\right)_T. \quad (3.53)$$

The claim follows by combining equations 3.46, 3.48 and 3.53. $\square$

Chapter 4

Optimal hedging

4.1 Introduction

In a complete market, under the assumption of no arbitrage, there exists a unique measure under which S is a martingale. In this case, all contingent claims can be replicated, the contingent claims are said to be attainable. This means that any contingent claim I can be represented as

$$I = I_0 + \int_0^T \vartheta_t dS_t$$

for some self-financing strategy ϑ, from the space of admissible strategies. A market driven by Brownian motions is complete when the number of sources of uncertainty is equal to the number of traded risky securities, the risky securities span the market. When there are more uncertainties than traded securities, the market is incomplete[1]. For example, this is usually the case in the models with stochastic volatility, but note that there exist complete stochastic volatility models. In the models with jumps, things become more involved. When the jump sizes are predictable, the market can be completed by introduction of another securities so that the risky securities again span the market. But once the securities can have jumps with unpredictable sizes, which is essentially as soon the security can have more than a single jump size, the markets will be always incomplete. Such markets cannot be completed with any number of traded securities.

[1]The market can be incomplete for many other reasons. For example, in the presence of transaction costs or short selling constraints the markets can become incomplete due to restriction of the class of portfolios one can construct.

In the incomplete market, under the assumption of no arbitrage, there are infinitely many martingale measures. We have seen in the previous chapter that there are infinitely many martingale measures already in the geometric or exponential Lévy model, i.e. we are in the incomplete market setting. Some of the contingent claims cannot be perfectly replicated and thus they also cannot be exactly priced with arbitrage arguments alone. There is an interval of arbitrage-free prices. One approach is then to study the range of arbitrage free prices, this is the concept of super-replication. Another approach is to introduce subjective criteria and then study the corresponding hedging strategies. We will study three such criteria in this chapter, namely local risk-minimization hedging, mean-variance hedging and q-optimal hedging.

In the local risk-minimization hedging problem, the hedging strategies are no longer self-financing but only mean-self-financing, while the cumulative trading gains are still equal to the terminal value of the contingent claim. The hedging strategies minimize locally the remaining risk, hence the name "local". We will discuss locally risk minimizing strategies in the next section.

The second approach is to keep the condition on the trading strategies to be self-financing, and trying to minimize, in some sense, the terminal hedging risk

$$I - I_0 - \int_0^T \vartheta_t dS_t.$$

In the mean-variance hedging, the optimal strategy minimizes the expected squared hedging risk, where the expectation refers to the real-world probability measure. We study the mean-variance hedging problem in section 4.3. Finally, in section 4.4, the more general problem of q-optimal hedging ($q > 1$), minimizing the expectation of the q-th power of the hedging risk, is studied in the martingale setting. From now on, we assume that I is not attainable, thus considering only non-trivial hedging cases.

4.2 Local risk-minimization hedging

In the incomplete market, some of the contingent claims cannot be perfectly hedged, that is, there does not exist a self-financing strategy such that the contingent claim can be written as a constant plus an integral of this self-financing strategy with respect to the price process. One approach is then to relax the constraint on the strategies to be self-financing, while still insisting on the fact that the terminal value of the trading by following the strategy has to equal to the value of the contingent claim. But as the strategies are no longer self-financing, there is a risk in writing a contingent claim. The local risk-minimization hedging aims to minimize this risk in a sequential way, locally. This problem was first studied by Föllmer and Sondermann (1986) in the martingale setting, and extended to semimartingale setting by Föllmer and Schweizer (1991). Let us introduce notation. Consider trading strategies of the form (ϑ, η), where ϑ and η describe the amounts invested into the stock and into the bond. It is assumed that ϑ is predictable and η is adapted. The value of this portfolio at time t is thus given by $U_t = \vartheta_t S_{t-} + \eta_t$,[2] and the cumulative trading gains in the stock are given by $G_t(\vartheta) = \int_0^t \vartheta_u dS_u$. The cost process is defined as

$$C_t = \vartheta_t S_{t-} + \eta_t - \int_0^t \vartheta_u dS_u = U_t - G_t(\vartheta). \tag{4.1}$$

For the strategies that are self-financing, the cost process is constant. A strategy (ϑ, η) is called mean-self-financing if the corresponding cost process $C = (C_t)_{0 \le t \le T}$ is a martingale. Note that any self-financing strategy is also mean-self-financing. Let us now introduce a contingent claim $I \in \mathbb{L}^2(\mathbb{P})$. A strategy is called admissible (with respect to I) if its value process has terminal value $U_T = I$, $\mathbb{P}$-a.s.[3]. The remaining risk is defined as $\mathbb{E}[(C_T - C_t)^2 | \mathcal{F}_t]$, and the strategy is called locally risk-minimizing if at any time this strategy minimizes the remaining risk. We first consider the case when the price process is a martingale.

[2]Without loss of generality, the bond value can be fixed to a constant 1.

[3]A contingent claim that can be written as a constant plus an integral of admissible self-financing strategy with respect to price process is called attainable. Using this terminology, the complete market is the one where any contingent claim is attainable.

4.2.1 Martingale setting

Let us assume that S is already a martingale under $\mathbb{P}$. Looking at the form of the value process and the cost process, it is clear that the strategy (ϑ, η) is mean-self-financing if and only if the value process is a square-integrable martingale. It can also be shown that admissible risk-minimizing strategy is mean-self-financing. These two facts imply that the value process of the local risk-minimizing strategy has to be a martingale. Consider the following decomposition of local martingales.

Definition 4.1. ***(Galtchouk-Kunita-Watanabe decomposition)*** *Let Y be a locally square-integrable local martingale. Then any local martingale M can be decomposed into following form*

$$M_t = M_0 + \int_0^t \xi_u dY_u + L_t, \tag{4.2}$$

where $(\xi_t)_{t\in[0,T]}$ is a square-integrable predictable strategy and L is a square-integrable local martingale such that $L_0 = 0$ and strongly orthogonal to Y.

For reference, cf. Jacod (1979), Theorem 4.27. The local risk-minimization hedging strategy is unique, given by the integrand in the above decomposition of the local martingale $(\mathbb{E}[I|\mathcal{F}_t])_{t\in[0,T]}$, cf. Theorem 2.1 in Schweizer (2001). The integrand ξ can be written as

$$\xi_t = \frac{\langle \mathbb{E}[I|\mathcal{F}_t], S_t\rangle}{\langle S_t, S_t\rangle}. \tag{4.3}$$

In view of the above, in the martingale setting, the local risk-minimization hedging problem can be seen as a projection problem: projecting a local martingale $\mathbb{E}[I|\mathcal{F}]$ on S.

We now find the explicit form the local risk-minimizing hedge for the class of models considered here. For the case when the price process is a martingale, we can find the optimal hedge even when we allow a correlation between the Brownian motions driving our price and volatility processes. Consider the stochastic model given by equation (2.17), and let us now assume that the Brownian motion B driving the price process and the Brownian motion B^V driving the volatility process have a correlation ρ, i.e. $d\langle B, B^V\rangle_t = \rho dt$. Let $U(t, S, V) = \mathbb{E}[I|\mathcal{F}_t]$. Using Itô's formula, we have

$$
\begin{aligned}
dU = {} & \frac{\partial U}{\partial t}dt + \frac{\partial U}{\partial S}dS + \frac{\partial U}{\partial V}dV + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}d\langle S, S\rangle + \frac{1}{2}\frac{\partial^2 U}{\partial V^2}d\langle V, V\rangle + \frac{\partial^2 U}{\partial U \partial V}d\langle S, V\rangle \\
& + (U(t, S, V_-) - U(t, S_-, V_-) - \triangle S\frac{\partial U}{\partial S}) \\
& + (U(t, S_-, V) - U(t, S_-, V_-) - \triangle V\frac{\partial U}{\partial V}).
\end{aligned}
$$

Because the discounted price process is a martingale, the drift is equal to zero and thus

$$
\begin{aligned}
dU = {} & \frac{\partial U}{\partial S}S\sigma dB + \frac{\partial U}{\partial V}\gamma dB^V \\
& + \int (U(t, S_-(1+\delta x), V_-) - U(t, S_-, V_-))N(dt, dx) \\
& + \int (U(t, S_-, V_-(1+\gamma x)) - U(t, S_-, V_-))N^V(dt, dx).
\end{aligned}
$$

Equation (4.3) implies

$$
\xi = \frac{\frac{\partial U}{\partial S}\sigma^2 + \frac{\partial U}{\partial V}\frac{\sigma\gamma\rho}{S} + \frac{1}{S}\int x(U(t, S_-(1+\delta x), V_-) - U(t, S_-, V_-))\nu(dx)}{\sigma^2 + \gamma^2\int x^2\nu(dx)}.
$$

This model can be considered as a generalization of the Bates model. The local risk-minimization hedging strategy for Bates model was determined in Hubalek and Sgarra (2007), however there is a mistake in their Proposition 3.2. In our notation, equation (3.14) for the local risk-minimization hedge reads

$$
\phi_t = \frac{V_t\frac{\partial U}{\partial S} + \frac{\partial U}{\partial V}\frac{V_t\rho\theta}{S_{t\_}} + \frac{1}{S_{t-}}\int (e^x - 1)(U(t, S_{t\_}e^x, V_t) - U(t, S_{t\_}, V_t))\nu(dx)}{V_t + \int (e^x - 1)^2\nu(dx)},
$$

where V is instantaneous variance. Another tractable stochastic volatility model where one can explicitly determine local risk-minimization hedge is Barndorff-Nielsen and Shephard model, see Cont, Tankov and Voltchkova (2007).

4.2.2 Semimartingale setting

The notion of risk-minimization introduced in the martingale setting by Föllmer and Sondermann (1986) cannot be readily applied to a semimartingale setting. Recall from (4.1) the equation for the cost process

$$
C_t = U_t - \int_0^t \vartheta_u dS = U_t - \int_0^t \vartheta_u dM_u - \int_0^t \vartheta_u dA_u.
$$

The problem is caused by the last term $\int_0^t \vartheta_u dA_u$ of which influence cannot be controlled in the equation for the remaining risk.

The notion of risk-minimization has been extended to semimartingale case by Schweizer (1991) by considering "infinitesimal perturbations" of the strategies. This is equivalent to the following definition.

Definition 4.2. *An admissible strategy (ϑ, η) is called locally risk-minimizing if the associated cost process C is a square-integrable martingale orthogonal to the martingale part of S under $\mathbb{P}$.*

Let Θ be the space given by Definition 3.7. The existence of an optimal strategy is equivalent to the existence of the following decomposition.

Definition 4.3. ***(Föllmer-Schweizer decomposition)***
The random variable $I \in L^2$ admits a Föllmer-Schweizer decomposition if I can be written as

$$I = I_0 + \int_0^T \xi_u^I dS_u + L_T^I \qquad \mathbb{P}\text{-}a.s., \tag{4.4}$$

such that $I_0 \in \mathbb{R}$ is a constant, $\xi^I \in \Theta$ is a strategy and $L^H = (L_t^H)_{0 \le t \le T}$ is a square-integrable martingale orthogonal to the martingale part of S.

Unfortunately, the Föllmer-Schweizer decomposition does not need to exist in the models with jumps. The simplest sufficient condition for existence of the Föllmer-Schweizer decomposition is when the mean-variance tradeoff process process $\hat{K}$ is uniformly bounded in $(t, w) \in [0, T] \times \Omega$. In the martingale setting, the Galtchouk-Kunita-Watanabe decomposition is the same as the Föllmer-Schweizer decomposition. When S is a semimartingale, one can start with Galtchouk-Kunita-Watanabe decomposition to derive an optimality equation, by which the optimal strategy is characterized. This is the approach taken in Schweizer (1991). In Föllmer and Schweizer (1991), the authors propose more natural way to solve the problem, by bringing it back to the martingale setting. They introduce the minimal martingale measure $\hat{\mathbb{P}}$, which is characterized by the following property: the martingale orthogonal to the martingale part of the price prices remains a martingale under the minimal martingale measure. Due to this property, using the Radon-Nikodým derivative of the minimal martingale measure and the Bayes formula, one can write

$$V_t^I = I_0 + \int_0^t \xi_u^I dS_u + L_t^I \tag{4.5}$$

as $V_t^I = \hat{\mathbb{E}}(I|\mathcal{F}_t)$, and also $I_0 = \hat{\mathbb{E}}(I)$. When the price process S is continuous, $\hat{\mathbb{P}}$ also preserves orthogonality in the sense that if L is orthogonal to the martingale part of S under $\mathbb{P}$, it is also orthogonal to S under $\hat{\mathbb{P}}$ because if L is orthogonal to S under $\hat{\mathbb{P}}$ then it is orthogonal to the martingale part of S under $\mathbb{P}$. This implies that one can find the Föllmer-Schweizer decomposition as the Galtchouk-Kunita-Watanabe decomposition under $\hat{\mathbb{P}}$. Thus, using the minimal martingale measure $\hat{\mathbb{P}}$, we are back in the martingale the martingale setting. However, this is usually no longer true when S is discontinuous. There seem to be some confusion about these concepts where the local risk-minimization strategies has been calculated by finding Galtchouk-Kunita-Watanabe decomposition under $\hat{\mathbb{P}}$ in the models involving jumps, cf. Cont and Tankov (2004), Section 10.4.3 or Hubalek and Sgarra (2007), Section 4.2, where the local risk-minimizing strategies are calculated in the jump-diffusion setting and correlated Bates model, respectively. We now calculate the optimal strategies in the model which includes both models mentioned above.

Consider the stochastic volatility model given by equation (2.17), and let us now assume that the Brownian motion B driving the price process and the Brownian motion B^V driving the volatility process have a correlation ρ, i.e. $d\langle B, B^V\rangle_t = \rho dt$. The minimal martingale measure that we determined in Chapter 3 remains the same even though the price process and volatility are now correlated.

Let $C(t, S, V) = \hat{\mathbb{E}}[I|\mathcal{F}_t]$. Using Itô's formula, we have

$$\begin{aligned} dC = {} & \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial V}dV + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}d\langle S, S\rangle + \frac{1}{2}\frac{\partial^2 C}{\partial V^2}d\langle V, V\rangle + \frac{\partial^2 C}{\partial C \partial V}d\langle S, V\rangle \\ & + (C(t, S, V_-) - C(t, S_-, V_-) - \triangle S\frac{\partial C}{\partial S}) \\ & + (C(t, S_-, V) - C(t, S_-, V_-) - \triangle V\frac{\partial C}{\partial V}), \end{aligned}$$

and because C is a $\hat{\mathbb{P}}$-martingale

$$\begin{aligned} dC &= \frac{\partial C}{\partial S} dS + \frac{\partial C}{\partial V} \gamma d\hat{B}^V \\ &+ \int (C(t, S_-(1+\delta x), V_-) - C(t, S_-, V_-) - S_- \delta x \frac{\partial C}{\partial S}) \hat{N}(dt, dx) \\ &+ \int (C(t, S_-, V_-(1+\gamma x)) - C(t, S_-, V_-)) N^V(dt, dx). \end{aligned} \tag{4.6}$$

Note that the compensated jump measure of V remains the same under $\hat{\mathbb{P}}$. Following (4.5), let

$$\begin{aligned} L_t^I &= V_t^I - I_0 - \int_0^t \xi_u^I dS_u \\ &= C(t, S, V) - C(0, S, V) - \int_0^t \xi_u^I dS_u. \end{aligned} \tag{4.7}$$

Using the characterizing property of $\hat{\mathbb{P}}$, we can now proceed in two ways. If L^I is from the Föllmer-Schweizer decomposition, we need L^I to be a

- $\mathbb{P}$-martingale,
- $\mathbb{P}$-orthogonal to the martingale part of S.

However, because L^I is a $\hat{\mathbb{P}}$-martingale $\hat{\mathbb{P}}$-orthogonal to S, by checking one of the above two properties, the other will follow automatically. This is another way of saying that, considering the above three properties of L^I being either $\mathbb{P}$-martingale or $\hat{\mathbb{P}}$-martingale $\hat{\mathbb{P}}$-orthogonal to S or $\mathbb{P}$-orthogonal to the martingale part of S, any two properties will imply the third. For example, we start by checking whether L^I is a $\mathbb{P}$-martingale. Combining equations (4.6) and (4.7) yields

$$\begin{aligned} dL_t^I &= \left(\frac{\partial C}{\partial S} - \xi^I \right) dS + \frac{\partial C}{\partial V} \gamma d\hat{B}^V \\ &+ \int (C(t, S_-(1+\delta x), V_-) - C(t, S_-, V_-) - S_- \delta x \frac{\partial C}{\partial S}) \hat{N}(dt, dx) \\ &+ \int (C(t, S_-, V_-(1+\gamma x)) - C(t, S_-, V_-)) N^V(dt, dx) \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\partial C}{\partial S} - \xi^I\right) dM + \frac{\partial C}{\partial V}\gamma dB^V \\
&\quad + \int (C(t, S_-(1+\delta x), V_-) - C(t, S_-, V_-) - S_-\delta x \frac{\partial C}{\partial S}) N(dt, dx) \\
&\quad + \int (C(t, S_-, V_-(1+\gamma x)) - C(t, S_-, V_-)) N^V(dt, dx) \\
&\quad + \left(\frac{\partial C}{\partial S} - \xi^I\right) \hat{\lambda} d\langle M\rangle + \frac{\partial C}{\partial V} S\gamma\rho\sigma\hat{\lambda} dt \\
&\quad + \int (C(t, S_-(1+\delta x), V_-) - C(t, S_-, V_-) - S_-\delta x \frac{\partial C}{\partial S}) \hat{h}\nu(dx) dt.
\end{aligned}
$$

Thus for L^I to be a $\mathbb{P}$-martingale, the following condition has to be satisfied:

$$
\begin{aligned}
&\left(\frac{\partial C}{\partial S} - \xi^I\right) S^2(\sigma^2 + \gamma^2 \int x^2 \nu(dx)) + \frac{\partial C}{\partial V} S\gamma\rho\sigma \\
&+ \int (C(t, S_-(1+\delta x), V_-) - C(t, S_-, V_-) - S_-\delta x \frac{\partial C}{\partial S}) \delta S x \nu(dx) = 0.
\end{aligned}
$$

From the last expression, the local risk minimization strategy is given by

$$
\xi^I = \frac{\partial C}{\partial S} + \frac{\frac{\partial C}{\partial V}\gamma\rho\sigma + \delta \int x(C(t, S_-(1+\delta x), V_-) - C(t, S_-, V_-) - S_-\delta x \frac{\partial C}{\partial S})\nu(dx)}{S_-(\sigma^2 + \delta^2 \int x^2 \nu(dx))}.
$$

Compare this with equation (4.18) in Hubalek and Sgarra (2007). In our notation, the optimal hedging strategy is given by

$$
\hat{\xi}^I = \frac{\sigma^2 \frac{\partial C}{\partial S} + \frac{\delta}{S_-} \int x(C(t, S_-(1+x), V) - C(t, S_-, V))\hat{\nu}(dx)}{\sigma^2 + \frac{\partial C}{\partial V}\frac{\gamma\rho\sigma^2}{S_-} + \delta^2 \int x^2 \hat{\nu}(dx)}.
$$

One can now calculate the quadratic covariation $\langle L^I, M\rangle$ under $\mathbb{P}$ using the hedging strategy $\hat{\xi}^I$ and equation (4.7) to see that this is not equal to zero. This means that L^I is not $\mathbb{P}$-orthogonal to M, i.e. L^I does not come from the Föllmer-Schweizer decomposition. Ultimately, $\hat{\xi}^I$ is not the local risk-minimization strategy.

4.3 Mean-variance hedging

One disadvantage of the local risk-minimization hedging might be the fact that the hedging strategies are only mean-self-financing. This means that during the life of the contingent claim, there might be intermediate expenses (or earnings). The alternative is to consider only hedging strategies that are self-financing but no longer admissible, that is $V_T \neq I$. This is the approach taken in the mean-variance hedging, which minimizes the hedging error at maturity in the mean-square sense.

Let $I \in L^2(\mathbb{P})$ be a $\mathcal{F}_T$-measurable random variable. The following optimization problem is called the mean-variance hedging problem:

$$\text{minimize} \quad \mathbb{E}\left[(I - G_T(\vartheta))^2\right] \tag{4.8}$$

over all reasonable trading strategies, or alternatively

$$\text{minimize} \quad \mathbb{E}\left[(I - c - G_T(\vartheta))^2\right]$$

where c varies over all possible initial amounts of capital. The question of which trading strategies one considers is important from several aspects. First, if the space of admissible strategies is too small some basic optimal strategies may not be included, if it is too big, there might be an arbitrage. Second, to guarantee existence of the solution to the mean-variance hedging problem, the space of admissible strategies has to be closed. To illustrate the complexity of the problem at hand, we will first review the main tools and approaches that are used to determine the mean-variance hedging strategy in the gradually more complex setting.

In the case when the price process S is already a martingale under the reference measure, the mean-variance hedging strategy is given by the integrand in the Galtchouk-Kunita-Watanabe decomposition (taken under the reference measure $\mathbb{P}$).

Using Galtchouk-Kunita-Watanabe decomposition, cf. equation (4.2), we have

$$\mathbb{E}\left[(I - c - G_T(\vartheta))^2\right] = \mathbb{E}\left[(G_T(\xi^I - \vartheta) + L_T^I)^2\right] + \mathbb{E}\left[(\mathbb{E}[I] - c)^2\right].$$

This shows that the optimal initial capital $c = \mathbb{E}[I]$, the mean-variance hedging strategy $\vartheta = \xi^I$ and the hedging error is equal to $\mathbb{E}[(L_T^I)^2]$. This also shows that in this case, the mean-variance and local risk-minimization hedging strategies coincide.

When the price process is no longer a martingale, the mean-variance hedging problem becomes more delicate. In the semimartingale setting, the problem was first tackled by imposing conditions on the model that implied equivalence of the minimal and the variance-optimal martingale measure. The most frequently used sufficient condition for that is when the mean-variance tradeoff process is deterministic, see Schweizer (1994), Pham, Rheinländer and Schweizer (1998) and Hubalek, Kallsen and Krawczyk (2006). The more general case, with the minimal and variance-optimal martingale measures being equal, has been studied by Hipp (1993) in a continuous model and by Wiese (1998) in a semimartingale model. The optimal hedge is found by using the Föllmer-Schweizer decomposition:

$$\vartheta^I = \xi^I + \hat{\lambda}(V^I - c - G_-(\vartheta^I)).$$

The process V^I is the one from equation (4.5), and ξ^I is the local risk-minimizing strategy, the integrand in the Föllmer-Schweizer decomposition. The optimal initial capital is the same as in the local risk-minimization hedging, given by $c = \hat{\mathbb{E}}[I]$.

The next step in the generalization is the case of a continuous semimartingale, with the minimal and the variance-optimal martingale measures being different. In this case, one needs to resort to the Galtchouk-Kunita-Watanabe decomposition under the variance-optimal martingale measure, see Rheinländer and Schweizer (1997) and the elegant approach by Gourieroux, Laurent and Pham (1998). By Lemma 1 of Schweizer (1996), the density of the variance optimal martingale measure can we written as

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \tilde{\mathbb{E}}\left[\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right] + \int_0^T \tilde{\zeta}_u dS_u$$

for some admissible $\tilde{\zeta}$. Define the process $\tilde{Z}$ as

$$\tilde{Z}_t = \mathbb{E}\left[\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\middle|\mathcal{F}_t\right] = \tilde{\mathbb{E}}[\tilde{Z}_T] + \int_0^t \tilde{\zeta}_u dS_u, \qquad 0 \le t \le T.$$

The optimal hedge is given by

$$\tilde{\vartheta}^I = \tilde{\xi}^I + \frac{\tilde{\zeta}}{\tilde{Z}_-}(\tilde{V}^I - c - G_-(\tilde{\vartheta}^I)),$$

where $\tilde{V}_t^I = \tilde{\mathbb{E}}[I|\mathcal{F}_t]$, $c = \tilde{\mathbb{E}}[I]$ and $\tilde{\xi}^I$ is the integrand in the Galtchouk-Kunita-Watanabe decomposition of I under $\tilde{\mathbb{P}}$, i.e.

$$\tilde{\xi}^I = \frac{\langle S, V^I \rangle_t^{\tilde{\mathbb{P}}}}{\langle S, S \rangle_t^{\tilde{\mathbb{P}}}}.$$

For thorough reviews of the quadratic hedging problem up to this point, see Schweizer (2001) and Pham (2000). The results for discontinuous semimartingale case are more limited. Arai (2005a) extended the work of Gourieroux, Laurent and Pham (1998) under assumption that $\tilde{\mathbb{P}}$ is equivalent martingale measure. This is the approach we adopt in this work. In the jump-diffusion model, Lim (2005) determines the optimal hedge by using backward stochastic differential equations. Černý and Kallsen (2007) provide a new characterization of the mean-variance hedging strategies in the general semimartingale setting. They introduce a new measure called the opportunity neutral measure, see Section 3.3.1. This measure neutralizes the effect of the stochastic mean-variance tradeoff process, which effectively brings the mean-variance hedging problem back to the case of deterministic mean-variance tradeoff process where the optimal hedge is given by the integrand in the Föllmer-Schweizer decomposition.

4.3.1 Semimartingale setting

Gourieroux, Laurent and Pham (1998) introduced a market extension and a change of numéraire to solve the mean-variance hedging problem in a diffusion model. This was subsequently generalized by Arai (2005a) to a more general setting allowing for jumps, under some additional assumptions. We adopt this approach here, however due to the chosen space Θ given in Definition 3.7, we do not need conditions (2) and (3) of Assumption 1 in Arai (2005a) to hold. Lemma 2.4 in Černý and Kallsen (2007) guarantees that the solution to (4.8) always exists, assuming $\mathbb{P}_e^2 \neq \varnothing$. Moreover, instead of condition (1) which assumes that the variance-optimal martingale measure is equivalent to the reference measure, we only assume that the variance-optimal martingale measure is a signed measure, non-zero almost surely.

Denote by Z the density of the variance-optimal martingale measure, assume that $\mathbb{P}\{\tilde{Z}_T = 0\} = 0$, and define

$$\tilde{Z}_t := \frac{\mathbb{E}[\,Z_T^2|\,\mathcal{F}_t]}{\mathbb{E}[\,Z_T|\,\mathcal{F}_t]} \qquad 0 \le t \le T. \tag{4.9}$$

Define a new probability measure $\mathbb{R}$ by

$$\frac{d\mathbb{R}}{d\mathbb{P}} = \frac{\tilde{Z}_T^2}{\tilde{Z}_0}, \tag{4.10}$$

and define the $\mathbb{R}^2$-valued process Y by

$$Y^0 := \tilde{Z}^{-1}, \quad Y^1 := S\tilde{Z}^{-1}.$$

A closer look at the proof of Proposition 8 in Rheinländer and Schweizer (1997) reveals that the assumption of a positive variance optimal martingale measure is in fact not necessary, and that more generally also in the case when the variance optimal martingale measure is a signed measure, the mean-variance hedging problem

$$\text{minimize} \quad \mathbb{E}[(I - G_T(\vartheta))^2] = \|I - G_T(\vartheta)\|^2_{L^2(\mathbb{P})} \quad \text{over all } \vartheta \in \Theta$$

is equivalent to the problem

$$\text{minimize} \quad \left\| \frac{I}{\tilde{Z}_T} - \int_0^T \psi_s dY_s \right\|_{L^2(\mathbb{R})} \qquad \text{over all } \psi \in L^2(Y, \mathbb{R}).$$

The relation between the integrands is given by

$$\vartheta := \psi^1 + \zeta \left\{ \int_0^- \psi dY - \psi^{tr} Y_- \right\}. \tag{4.11}$$

Since $I/\tilde{Z}_T \in L^2(\mathbb{R})$, and since Y is a square-integrable $\mathbb{R}$-local martingale, there exists a Galtchouk-Kunita-Watanabe decomposition of $I/\tilde{Z}_T$ on Y under $\mathbb{R}$

$$\frac{I}{\tilde{Z}_T} = \mathbb{E}_{\mathbb{R}}\left[\frac{I}{\tilde{Z}_T}\right] + \int_0^T \psi_u dY_u + M_T, \tag{4.12}$$

where $\psi \in L^2(Y, \mathbb{R})$, and M is a square-integrable $\mathbb{R}$-martingale, $\mathbb{R}$-orthogonal to Y. Thus, the problem is transformed into the martingale case, where the optimal hedging strategy in given by the integrand of the Galtchouk-Kunita-Watanabe decomposition. In what follows, we use the superscript $\mathbb{R}$ to denote quantities related to the measure $\mathbb{R}$.

The problem of finding the solution to the mean-variance hedging problem is thus equivalent to finding the integrand $\tilde{\psi}^I$ in the Galtchouk-Kunita-Watanabe decomposition given in (4.12).

In order to determine the variance-optimal hedging strategies, we need to find the dynamics of the volatility process V under $\tilde{\mathbb{P}}$. Thus, we have to find processes F and f that satisfy

$$\mathcal{E}\left(\int F_u dB_u^V + \iint_{\mathbb{R}} f(u,x) N^V(du,dx)\right)_T = \frac{\exp(-\hat{K}_T)}{\mathbb{E}\left[\exp(-\hat{K}_T)\right]}. \tag{4.13}$$

and whose existence is justified by the martingale representation property. In the setting of stochastic volatility models driven by diffusion processes the solution to equation (4.13) was obtained by Laurent and Pham (1999) using a dynamical programming approach, and by martingale techniques by Biagini et al. (2000) and Hobson (2004). Using a Markov framework assumption, we can generalize these results to our setting.Using the model assumptions, namely the Markov property and the process $\hat{K}$ being a function of V alone, the following generalizes these results to our setting.

Proposition 4.4. *Define the process $R = R(t, V_t)$ by*

$$R(t, V_t) = \mathbb{E}\left[\exp\left(\hat{K}_t - \hat{K}_T\right) \Big| \mathcal{F}_t\right]. \tag{4.14}$$

Assume $R(t, v) \in C^{1,2}$. Then

$$\mathcal{E}\left(\int F_u dB_u^V + \iint_{\mathbb{R}} f(u,x) N^V(du,dx)\right)_T = \frac{\exp(-\hat{K}_T)}{\mathbb{E}\left[\exp(-\hat{K}_T)\right]}$$

if and only if the processes F and f are given by

$$F_t = \frac{\gamma(t, V_{t-}) \frac{\partial R}{\partial v}(t, V_{t-})}{R(t, V_{t-})}, \tag{4.15}$$

$$f(t,x) = \frac{\Delta R(t, V_t)}{R(t, V_{t-})}. \tag{4.16}$$

Proof. Define the martingale D by $D_t = \mathbb{E}[\exp(-\hat{K}_T)|\mathcal{F}_t]$. By the martingale representation property, there exist processes ϕ and ψ, such that

$$D_t = D_0 + \int_0^t \phi_u dB_u^V + \int_0^t \int_{\mathbb{R}} \psi(u,x) N^V(du,dx). \tag{4.17}$$

Therefore,

$$\begin{aligned} D_t &= \exp(-\hat{K}_t)\mathbb{E}\left[\exp\left(-\int_t^T \hat{\lambda}_u(u, V_{u-}) dA_u\right) \Big| \mathcal{F}_t\right] \\ &= \exp(-\hat{K}_t) R(t, V_t) \\ &= D_0 + \int_0^t R(t, V_t) d(\exp(-\hat{K}_t)) + \int_0^t \exp(-\hat{K}_t) d(R(t, V_t)). \end{aligned}$$

Here we have used the fact that $\hat{K}$ is a continuous process of finite variation. Since D is a martingale, the terms of finite variation vanish, because they are also predictable, and it follows that

$$\begin{aligned} dD_t &= \exp(-\hat{K}_t)\left[\gamma(t,V_{t-})\frac{\partial R}{\partial v}(t,V_{t-})\left(dB_t^V + \int_{\mathbb{R}} xN^V(dt,dx)\right)\right. \\ &\quad \left. + \int_{\mathbb{R}} \big(R(t,V_t) - R(t,V_{t-}) - \gamma(t,V_{t-})x\frac{\partial R}{\partial v}(t,V_{t-})\big)N^V(dt,dx)\right] \\ &= \exp(-\hat{K}_t)\left[\gamma(t,V_{t-})\frac{\partial R}{\partial v}(t,V_{t-})dB_t^V\right. \\ &\quad \left. + \int_{\mathbb{R}} \big(R(t,V_{t-} + \gamma(t,V_{t-})x) - R(t,V_{t-})\big)N^V(dt,dx)\right] \end{aligned}$$

Equation (4.17) implies that

$$\begin{aligned} \phi_t &= \exp(-\hat{K}_t)\gamma(t,V_{t-})\frac{\partial R}{\partial v}(t,V_{t-}), \\ \psi(t,x) &= \exp(-\hat{K}_t)\big(R(t,V_{t-} + \gamma(t,V_{t-})x) - R(t,V_{t-})\big). \end{aligned}$$

Rewriting now equation (4.17) as a stochastic exponential

$$D_T = D_0\mathcal{E}\left(\int \frac{1}{R_{u-}}\left(\phi_u dB_u^V + \int_{\mathbb{R}} \psi(u,x)N^V(du,dx)\right)\right)_T,$$

yields

$$\begin{aligned} F_t = \phi_t/D_{t-} &= \frac{\gamma(t,V_{t-})\frac{\partial R}{\partial v}(t,V_{t-})}{R(t,V_{t-})}, \\ f(t,x) = \psi(t,x)/D_{t-} &= \frac{\Delta R(t,V_t)}{R(t,V_{t-})}. \end{aligned}$$

□

Define the $\tilde{\mathbb{P}}$-martingales $\tilde{B}^V$ and $\tilde{N}^V$ by

$$d\tilde{B}_t^V = dB_t^V - F_t dt, \tag{4.18}$$

$$d\tilde{N}_t^V = \int_{\mathbb{R}} xN^V(dt,dx) - \int_{\mathbb{R}} xf(t,x)\nu^V(dx)du. \tag{4.19}$$

The volatility process then satisfies

$$\begin{aligned} dV_t &= g(t,V_{t-})dt + \gamma(t,V_{t-})dX_t^V \\ &= \tilde{g}(t,V_{t-})dt + \gamma(t,V_{t-})(d\tilde{B}_t^V + d\tilde{N}_t^V), \end{aligned}$$

where

$$\tilde{g}(t,V_{t-}) = g(t,V_{t-}) + \gamma(t,V_{t-})\big(a^V + F_t + \int_{\mathbb{R}} xf(t,x)\nu^V(dx)\big).$$

Proposition 4.5. *The Radon-Nikodým derivative of* $\mathbb{R}$ *with respect to* $\tilde{\mathbb{P}}$ *is given by*

$$\frac{d\mathbb{R}}{d\tilde{\mathbb{P}}} = \mathcal{E}\left(\int \hat{G}_u d\hat{B}_u + \iint_{\mathbb{R}} \hat{h}(u,x)\hat{N}(du,dx)\right)_T.$$

Proof. For any equivalent martingale measure $\mathbb{Q}$ with square integrable density $\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right] = \frac{1}{\mathbb{E}[\mathcal{E}(-\int \hat{\lambda} dS)_T]}$ and so

$$\tilde{Z}_0 = \tilde{\mathbb{E}}\left[\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right] = \frac{1}{\mathbb{E}[\mathcal{E}(-\int \hat{\lambda} dS)_T]}.$$

Thus, by definition of $\mathbb{R}$,

$$\frac{d\mathbb{R}}{d\tilde{\mathbb{P}}} = \frac{\tilde{Z}_T}{\tilde{Z}_0} = \mathcal{E}\left(-\int \hat{\lambda}_u dS_u\right)_T$$

and using the form of $\hat{G}$ and $\hat{h}$ yields the proposition. □

Note that the transform to the measure $\mathbb{R}$ does not affect the volatility process V. We have

$$dV_t = \tilde{g}(t, V_{t-})dt + \gamma(t, V_{t-})(dB_t^{V,\mathbb{R}} + dN_t^{V,\mathbb{R}}),$$

where $B^{V,\mathbb{R}}$ and $N^{V,\mathbb{R}}$ are $\mathbb{R}$-martingales defined by the measure transformation of the $\tilde{\mathbb{P}}$-martingales given by equations (4.18) and (4.19). It remains to find the dynamics of Y under $\mathbb{R}$.

Proposition 4.6. *The* $\mathbb{R}$*-dynamics of* $Y = (Y^0, Y^1) = \left(\frac{1}{\tilde{Z}}, \frac{S}{\tilde{Z}}\right)$ *are given by*

$$dY_t^0 = Y_{t-}^0 \hat{\lambda} S\left\{\sigma dB_t^{\mathbb{R}} + \delta \int_{\mathbb{R}} \frac{x}{1-\delta\hat{\lambda}Sx} N^{\mathbb{R}}(du,dx)\right\} \tag{4.20}$$

$$dY_t^1 = Y_{t-}^1 (1+\hat{\lambda}S)\left\{\sigma dB_t^{\mathbb{R}} + \delta \int_{\mathbb{R}} \frac{x}{1-\delta\hat{\lambda}Sx} N^{\mathbb{R}}(du,dx)\right\} \tag{4.21}$$

where $B^{\mathbb{R}}$ *is a* $\mathbb{R}$*-Wiener process and the compensator of* J *is given by* $\nu_t^{\mathbb{R}}(dx) = (1+\delta x\hat{\lambda}S)^2\nu(dx)$.

Proof. In the first step, we evaluate Y^0:

$$dY^0 = d\left(\frac{1}{\tilde{Z}}\right) = -\frac{d\tilde{Z}}{\tilde{Z}_-^2} + \frac{d[\tilde{Z},\tilde{Z}]^c}{\tilde{Z}_-^3} + \frac{1}{\tilde{Z}} - \frac{1}{\tilde{Z}_-} + \frac{\Delta\tilde{Z}}{\tilde{Z}_-^2}.$$

Using $d\tilde{Z} = -\hat{\lambda}\tilde{Z}_{-}dS$, we have

$$\begin{aligned}
dY^0 &= \frac{1}{\tilde{Z}_{-}}\left(\hat{\lambda}dS + \hat{\lambda}^2 d[S,S]^c + \frac{1}{1-\hat{\lambda}\Delta S} - 1 - \hat{\lambda}\Delta S\right) \\
&= \frac{1}{\tilde{Z}_{-}}\left(-\hat{G}d\hat{B} - \int_{\mathbb{R}} \hat{h}(t,x)\hat{N}(dt,dx) + \hat{G}^2 dt \right.\\
&\qquad \left. + \int_{\mathbb{R}}\left(\frac{1}{1+\hat{h}(t,x)} - 1 + \hat{h}(t,x)\right) Q(dt,dx)\right) \\
&= \frac{1}{\tilde{Z}_{-}}\left(-\hat{G}d\hat{B} + \hat{G}^2 dt + \int_{\mathbb{R}}\left(\frac{1}{1+\hat{h}(t,x)} - 1\right)\hat{N}(dt,dx)\right.\\
&\qquad \left. + \int_{\mathbb{R}}\left(\frac{1}{1+\hat{h}(t,x)} - 1 + \hat{h}(t,x)\right)\hat{\nu}(dt,dx)\right). \qquad (4.22)
\end{aligned}$$

By the product rule

$$dY_t^1 = d\left(\frac{S_t}{\tilde{Z}_t}\right) = S_{t-}d\left(\frac{1}{\tilde{Z}_t}\right) + \frac{1}{\tilde{Z}_{t-}}dS_t + d[S, 1/\tilde{Z}]_t. \qquad (4.23)$$

Combining equations (4.22) and (4.23) yields

$$\begin{aligned}
dY^1 = Y_{-}^1 &\left\{-\hat{G}d\hat{B} + \hat{G}^2 dt + \int_{\mathbb{R}}\left(\frac{1}{1+\hat{h}(t,x)} - 1\right)\hat{N}(dt,dx)\right.\\
&+ \int_{\mathbb{R}}\left(\frac{1}{1+\hat{h}(t,x)} - 1 + \hat{h}(t,x)\right)\hat{\nu}(dt,dx) + \delta(t,V_{t-})\int_{\mathbb{R}} x\hat{N}(dt,dx)\\
&- [\hat{G}d\hat{B}, \sigma(t,V_{t-})d\hat{B}] + \sigma(t,V_{t-})d\hat{B}\\
&\left.\left[\int_{\mathbb{R}}\left(\frac{1}{1+\hat{h}(t,x)} - 1\right)\hat{N}(dt,dx), \int_{\mathbb{R}}\delta(t,V_{t-})x\hat{N}(dt,dx)\right]\right\}\\
= Y_{-}^1 &\left\{(\sigma(t,V_{t-}) - \hat{G})d\hat{B} + \int_{\mathbb{R}}\left(\frac{\delta(t,V_{t-})x - \hat{h}(t,x)}{1+\hat{h}(t,x)}\right)\hat{N}(dt,dx)\right.\\
&\left. + (\hat{G}^2 - \hat{G}\sigma(t,V_{t-}))dt + \int_{\mathbb{R}}\left(\frac{\hat{h}(t,x)^2 - \hat{h}(t,x)\delta(t,V_{t-})x}{1+\hat{h}(t,x)}\right)\hat{\nu}(dt,dx)\right\}.
\end{aligned}$$

Proposition 4.5 now implies that Y is local $\mathbb{R}$-martingale and that it takes the form given in the statement of the theorem. □

The Markovian structure of our model implies that finding a Galtchouk-Kunita-Watanabe decomposition of $I/\tilde{Z}_T$ under $\mathbb{R}$ is equivalent to finding a solution to PIDE. Let

$$\begin{aligned}
u(t, Y_t^0, Y_t^1, V_t) &= \mathbb{E}_{\mathbb{R}}[I/\tilde{Z}_T | \{1/\tilde{Z}_u, (S/\tilde{Z})_u, V_u : u \le t\}]\\
&= \mathbb{E}_{\mathbb{R}}[w(Y_T^0, Y_T^1, V_T) | \{1/\tilde{Z}_u, (S/\tilde{Z})_u, V_u : u \le t\}].
\end{aligned}$$

Note that due to the fact that the numéraire $\tilde{Z}$ is used not only to define the measure $\mathbb{R}$ but also to extend the space of tradeable assets, we have another dimension to deal with. Assuming regularity of both model parameters and the function $u(t,z,y,v)$ implies

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \tilde{g}(t,v)\frac{\partial u}{\partial v} + \frac{1}{2}\left(\gamma^2\frac{\partial^2 u}{\partial v^2} + (y\sigma(1+\hat{\lambda}S))^2\frac{\partial^2 u}{\partial y^2} + (z\sigma\hat{\lambda}S)^2\frac{\partial^2 u}{\partial z^2}\right)\\
&- zy\sigma^2\hat{\lambda}S(1+\hat{\lambda}S)\frac{\partial^2 u}{\partial z\partial y} + \int\left\{u(t,z,y,v+\gamma x) - u(t,z,y,v) - \gamma x\frac{\partial u}{\partial v}\right\}\nu_t^{V,\mathbb{R}}(dx)\\
&+\int\left\{u\left(t,\frac{z}{1-\delta x\hat{\lambda}S},\frac{y+\delta xy}{1-\delta x\hat{\lambda}S},v\right) - u(t,z,y,v)\right.\\
&\qquad\left. - \frac{\delta x\hat{\lambda}Sz}{1-\delta x\hat{\lambda}S}\frac{\partial u}{\partial z} - \frac{\delta x(1+\hat{\lambda}S)y}{1-\delta x\hat{\lambda}S}\frac{\partial u}{\partial y}\right\}\nu_t^{\mathbb{R}}(dx) = 0, \qquad (4.24)
\end{aligned}$$

with terminal condition

$$u(T,z,y,v) = w(z,y,v). \qquad (4.25)$$

By the Galtchouk-Kunita-Watanabe decomposition

$$\begin{aligned}
u_T - u_0 &= \int_0^T \psi_t^0 dY_t^0 + \psi_t^1 dY_t^1 + L_T\\
&= \int_0^T \sigma(S_{t-}\lambda_t Y_{t-}^0\psi_t^0 + (1+S_{t-}\lambda_t)Y_{t-}^1\psi_t^1)dB_t^{\mathbb{R}}\\
&\quad + \int_0^T\int_{\mathbb{R}}\frac{\delta x}{1-\delta x\hat{\lambda}S}(S\lambda Y_{t-}^0\psi_t^0 + (1+S\lambda)Y_{t-}^1\psi_t^1))N^{\mathbb{R}}(dt,dx) + L_T,
\end{aligned} \qquad (4.26)$$

and from Itô's formula

$$\begin{aligned}
u_T - u_0 = &\int_0^T \sigma\left(\hat{\lambda}SY_{t-}^0\frac{\partial u}{\partial x} + (1+\hat{\lambda}S)Y_{t-}^1\frac{\partial u}{\partial y}\right)dB_t^{\mathbb{R}} + \int_0^T\frac{\partial u}{\partial v}dB^{V,\mathbb{R}}\\
&+\int_0^T\int_{\mathbb{R}}\left[u(t,x+\frac{\delta xz\hat{\lambda}S}{1-\delta x\hat{\lambda}S},y+\frac{\delta yz(1+\hat{\lambda}S)}{1-\delta x\hat{\lambda}S},v) - u(t,z,y,v)\right]N^{\mathbb{R}}(dx,dt)\\
&+\int_0^T\int_{\mathbb{R}}[u(t,z,y,v+\gamma x) - u(t,z,y,v)]N^{V,\mathbb{R}}(dx,dt).
\end{aligned}$$

Using sharp brackets properties and the fact that L is orthogonal to each component of Y

$$d\langle Y^0,u\rangle_t = \psi^0 d\langle Y^0,Y^0\rangle_t + \psi^1 d\langle Y^0,Y^1\rangle_t, \qquad (4.27)$$

$$d\langle Y^1,u\rangle_t = \psi^0 d\langle Y^1,Y^0\rangle_t + \psi^1 d\langle Y^1,Y^1\rangle_t. \qquad (4.28)$$

However, equations (4.27) and (4.28) are linearly dependent, which can be seen after noting that

$$\frac{dY_t^0}{Y_{t-}^0 \hat{\lambda} S} = \frac{dY_t^1}{Y_{t-}^1 (1 + \hat{\lambda} S)}, \tag{4.29}$$

and thus there are infinitely many pairs of (ψ^0, ψ^1) satisfying (4.27) and (4.28). Using (4.29), equation (4.26) can be written as

$$u_T - u_0 = \int_0^T \psi_t^* dY_t^0 + L_T$$

where

$$\psi^* = \psi^0 + \frac{Y^1(1 + \hat{\lambda} S)}{Y^0 \hat{\lambda} S} \psi^1. \tag{4.30}$$

The integrand in the Galtchouk-Kunita-Watanabe decomposition is given by

$$\begin{aligned}
\psi_t^* &= \frac{d\langle Y^0, u\rangle_t}{d\langle Y^0, Y^0\rangle_t} \\
&= \frac{1}{\sigma^2 + \delta^2 \int \left(\frac{x}{1-\delta x \hat{\lambda} S}\right)^2 \nu_t^{\mathbb{R}}(dx)} \left(\sigma^2 \frac{\partial u}{\partial x} + \sigma^2 \frac{Y^1(1+\hat{\lambda} S)}{Y^0 \hat{\lambda} S} \frac{\partial u}{\partial y} \right. \\
&\quad \left. - \frac{\delta}{Y^0 \hat{\lambda} S} \int \frac{x}{1 - \delta x \hat{\lambda} S} \left[u(t, x + \frac{\delta x z \hat{\lambda} S}{1 - \delta x \hat{\lambda} S}, y + \frac{\delta y z (1 + \hat{\lambda} S)}{1 - \delta x \hat{\lambda} S}, v) - u(t, z, y, v) \right] \nu_t^{\mathbb{R}}(dx) \right).
\end{aligned}$$

The variance optimal hedging strategy by equation (4.11) can now be written as

$$\begin{aligned}
\vartheta &= \psi^1 + \zeta \left(\int_0^- \psi dY - \psi^{tr} Y_- \right) && (4.31) \\
&= \hat{\lambda} \psi^* + \zeta \int_0^- \psi^* dY^0. && (4.32)
\end{aligned}$$

When the amount to set up the initial hedge is the "approximation price" given by $\tilde{\mathbb{E}}[I]$, the mean variance hedging strategy is equal to

$$\tilde{\vartheta} = \vartheta + \zeta \mathbb{E}_{\mathbb{R}} \left[\frac{I}{\tilde{Z}_T} \right].$$

4.3.2 Equivalent variance-optimal martingale measure

In this section we assume assume that the variance-optimal martingale measure is equivalent to the reference measure, and we discuss two problems related to the decomposition given by (4.12).

In the case of equivalent variance-optimal martingale measure, we can rewrite equation (4.9) as follows:

$$\tilde{Z}_t := \tilde{\mathbb{E}}\left[\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\middle|\mathcal{F}_t\right] = \tilde{Z}_0 + G_t(\tilde{\zeta}_t),$$

where

$$\tilde{Z}_0 = \tilde{\mathbb{E}}\left[\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right], \qquad \tilde{\zeta}_t = -\tilde{Z}_0\hat{\lambda}_t\mathcal{E}\left(-\int_0^t \hat{\lambda}_u dS_u\right)_- .$$

Equation (4.12) can now be readily transformed to the following one

$$I = \tilde{\mathbb{E}}[I] + G_T(\tilde{\eta}^I) + N_T^I, \tag{4.33}$$

where

$$\tilde{\eta}_t^I := \tilde{\mathbb{E}}[I]\tilde{Z}_0^{-1}\tilde{\zeta}_t + \tilde{\vartheta}_t^I + M_{t-}\tilde{\zeta}_t$$

and

$$N^I := \int \tilde{Z}_{u-}dM_u + [\tilde{Z}, M].$$

The first problem we discuss is whether in our model the solution to the mean-variance hedging is the same in both continuous and discontinuous case. This is to ask whether the decomposition (4.33) is a Galtchouk-Kunita-Watanabe decomposition. Using equation (4.33), we see that for (4.33) to be the Galtchouk-Kunita-Watanabe decomposition, the process N^I needs to be $\tilde{\mathbb{P}}$-orthogonal to S. Arai (2005a) outlines why (4.33) is not a Galtchouk-Kunita-Watanabe decomposition in general, and in Arai (2005c) illustrates this disparity in a one-dimensional jump diffusion model. We will proceed similarly using our model.

Under the variance-optimal martingale measure $\tilde{\mathbb{P}}$, we have

$$S_t = S_0 + \int_0^t S_{u-}(\sigma_u d\tilde{B}_u + \delta_u d\tilde{N}_u).$$

By the martingale representation theorem, the $\tilde{\mathbb{P}}$-martingale M from (4.12) is represented as

$$M_t = \int_0^t v_u^M d\tilde{B}_u + \int_0^t\int_{\mathbb{R}} w^M(u,x)\tilde{N}(du,dx) + \tilde{U}_t^M$$

for some predictable processes v^M and w^M, and the process $\tilde{U}^M$ which is orthogonal to both continuous and discontinuous part of S. Because M is also $\tilde{\mathbb{P}}$-orthogonal to

S, the quadratic covariation $[S, M]$ needs to be a local $\tilde{\mathbb{P}}$-martingale, which implies the following orthogonality condition

$$\int_0^t \sigma_u v_u^M du + \int_0^t \int_{\mathbb{R}} \delta_u x w^M(u,x) \tilde{\nu}(dx) du = 0.$$

Thus we have

$$\begin{aligned}
[S, M]_t &= \int_0^t S_{u-} \sigma_u v_u^M du + \int_0^t \int_{\mathbb{R}} S_{u-} \delta_u x w^M(u,x) J(du, dx) \\
&= \int_0^t \int_{\mathbb{R}} S_{u-} \delta_u x w^M(u,x) \tilde{N}(du, dx). \qquad (4.34)
\end{aligned}$$

Using the above result, we have

$$\begin{aligned}
N_t^I &= \int_0^t \tilde{Z}_{u-} dM_u + [\tilde{Z}, M]_t = \int_0^t \tilde{Z}_{u-} dM_u + \int_0^t \tilde{\zeta}_u d[S, M]_u \\
&= \int_0^t \tilde{Z}_{u-} v_u^M d\tilde{B}_u + \int_0^t \tilde{Z}_{u-} d\tilde{U}_u^M \\
&\quad + \int_0^t \int_{\mathbb{R}} \big(\tilde{Z}_{u-} w^M(u,x) + S_{u-} \delta_u x w^M(u,x) \tilde{\zeta}_u \big) \tilde{N}(du, dx).
\end{aligned}$$

The last line shows that N^I is a local $\tilde{\mathbb{P}}$-martingale. Finally

$$[N^I, S]_t = \int_0^t \tilde{Z}_{u-} d[S, M]_u + [S, [\tilde{Z}, M]]_t, \qquad (4.35)$$

and because $[S, M]$ is always a local $\tilde{\mathbb{P}}$-martingale, as we have seen in (4.34), we calculate the process $[S, [\tilde{Z}, M]]$ only:

$$\begin{aligned}
[S, [\tilde{Z}, M]] &= [S, \int_0^t \int_{\mathbb{R}} S_{u-} \delta_u x w^M(u,x) \tilde{\zeta}_u \tilde{N}(du, dx)] \\
&= \int_0^t \int_{\mathbb{R}} S_{u-}^2 \delta_u^2 x^2 w^M(u,x) \tilde{\zeta}_u J(du, dx).
\end{aligned}$$

Thus $[S, [\tilde{Z}, M]]_t$, and therefore also $[N^I, S]_t$, is not a local $\tilde{\mathbb{P}}$-martingale, implying that N^I is not $\tilde{\mathbb{P}}$-orthogonal to S.

As the second problem, we discuss the case when mean-variance tradeoff process is deterministic. Following Arai (2005b), we show that the new decomposition (4.33) becomes the Föllmer-Schweizer decomposition. Recall that when the mean-variance tradeoff process is deterministic, the minimal martingale measure and the variance-optimal martingale measures are equivalent, which implies that $\tilde{\zeta} = -\hat{\lambda} \tilde{Z}_-$. Let us reming how the measure transform affects the pure jump martingale process:

$$\begin{aligned}\tilde{N}(dt,dx) &= J(dt,dx) - \hat{\nu}(dx)dt \\ &= J(dt,dx) - \big(\hat{h}(t,x)+1\big)\nu(dx)dt \\ &= N(dt,dx) - \hat{h}(t,x)\nu(dx)dt.\end{aligned}$$

Therefore the second term in (4.35) can we rewritten as

$$\begin{aligned}[S,[\tilde{Z},M]] &= -\int_0^t\int_{\mathbb{R}} S_{u-}^2\delta_u^2 x^2 w^M(u,x)\hat{\lambda}_u\tilde{Z}_{u-}J(du,dx) \\ &= -\int_0^t\int_{\mathbb{R}} S_{u-}^2\delta_u^2 x^2 w^M(u,x)\hat{\lambda}_u\tilde{Z}_{u-}\big(\tilde{N}(du,dx) - \bar{\nu}(dx)du\big). \qquad (4.36)\end{aligned}$$

Using equation (3.5), the first term in (4.35) can be written as

$$\begin{aligned}\int_0^t \tilde{Z}_{u-}d[S,M]_u &= \int_0^t\int_{\mathbb{R}} \tilde{Z}_{u-}S_{u-}\delta_u x w^M(u,x)\tilde{N}(du,dx) \\ = \int_0^t\int_{\mathbb{R}} \tilde{Z}_{u-}S_{u-}\delta_u x w^M(u,x)N(du,dx) &+ \int_0^t\int_{\mathbb{R}} S_{u-}^2\delta_u^2 x^2 w^M(u,x)\hat{\lambda}_u\tilde{Z}_{u-}\nu(dx)du.\end{aligned} \qquad (4.37)$$

Summing (4.36) and (4.37), we have a local $\mathbb{P}$-martingale

$$[N^I,S]_t = \int_0^t\int_{\mathbb{R}} \tilde{Z}_{u-}S_{u-}\delta_u x w^M(u,x)(1 - S_{u-}\delta_u\hat{\lambda}_u x)N(du,dx).$$

Arai (2002) shows in Proposition 4.2 that N^I in this case is a $\mathbb{P}$-martingale, implying that the new decomposition in (4.33) is a Föllmer-Schweizer decomposition under $\mathbb{P}$. He further remarks that (4.33) is then also a Galtchouk-Kunita-Watanabe decomposition under $\hat{\mathbb{P}}$. However, this is equivalent to saying that the Galtchouk-Kunita-Watanabe decomposition under $\hat{\mathbb{P}}$ can be calculated as the Föllmer-Schweizer decomposition under $\mathbb{P}$, or, that a martingale process orthogonal to the martingale part of S under $\mathbb{P}$ remains orthogonal to S under the minimal martingale measure $\hat{\mathbb{P}}$. This question was investigated in the multidimensional jump diffusion setting in Wiesenberg (1998). For the one-dimensional jump diffusion model considered in Arai (2002), much stronger condition than only deterministic mean-variance tradeoff are needed. One example could be when the mean-variance tradeoff process is constant and the process M is orthogonal to both components of S.

4.4 The p-optimal hedging

In this section we assume that $I \in L^p(\mathbb{P})$ is a $\mathcal{F}_T$-measurable random variable. The investor in the incomplete market trying to hedge the payoff I in the L^p-sense needs to solve the following problem

$$\text{minimize} \quad \mathbb{E}\left[(I - G_T(\vartheta))^p\right] \tag{4.38}$$

over all reasonable trading strategies ϑ. The above problem is called the p-optimal hedging problem, and the particular case for $p = 2$, the mean-variance hedging, has been studied in the previous section. Similarly to the mean-variance hedging, p-optimal hedging can be considered as a projection problem: L^p projection of $I \in L^p(\mathbb{P})$ onto the space of stochastic integral with respect to S. We consider only the simplest case when S is already a martingale. Recall that in that case, the mean-variance hedging strategy is given by the integrand in the Galtchouk-Kunita-Watanabe decomposition. The following theorem shows how that decomposition generalizes for the projection in L^p spaces.

Theorem 4.7. *(Luenberger (1969), Theorem 5.8.1)*
Let $I \in L^p(\mathbb{P})$ be an $\mathcal{F}_T$-measurable random variable. Then there exists $\vartheta \in \theta^p$ and $L_T \in L^p(\mathbb{P})$ such that

$$I = \mathbb{E}(I) + G_T(\vartheta) + L_T,$$

and such that for any $\psi \in \theta^p$

$$\mathbb{E}[\mathrm{sgn}(\mathbb{E}(I) + L_T)|\mathbb{E}(I) + L_T|^{p-1} G_T(\psi)] = 0 \tag{4.39}$$

and $\mathbb{E}(L_T) = 0$. The p-optimal hedging strategy is unique, given by ϑ.

Proof. cf. Arai (2006), proof of Theorem 4.1. □

Our aim is now to specify the optimal hedge ϑ more explicitly in certain cases. We start with a geometric Lévy process. Let the dynamics of S under $\mathbb{P}$ be given by

$$dS = S_- dX$$

where X is a Lévy process with Lévy measure ν, constant diffusion coefficient σ and zero drift, as we assume that S is a martingale under $\mathbb{P}$.

Proposition 4.8. *Let $V(t, S_{t-}) = \mathbb{E}(I|\mathcal{F}_t)$ and assume $V \in \mathcal{C}^{1,2}$. The optimal hedge, the amount of underlying ϑ to be hold at time t in the p-optimal hedging problem, solves the following integral equation*

$$(p-1)G_t\sigma + \int_{\mathbb{R}} g_{p-1}(h(t,x))x\nu(dx) = 0 \tag{4.40}$$

where

$$G_t = \frac{(\frac{\partial V}{\partial S} - \vartheta)S_-\sigma}{\mathcal{E}\left(\int GdB + \int\int hN\right)_-}, \qquad h(t,x) = \frac{V(t, S_-(1+x)) - V(t, S_-) - S_-\vartheta x}{\mathcal{E}\left(\int GdB + \int\int hN\right)_-}. \tag{4.41}$$

Proof. First note that $V(T, S_T) = I$. By Itô's formula

$$V(T,S_T) - V(0,S_0) = \int_0^T \frac{\partial V}{\partial S} S_-\sigma dB + \int_0^T \int_{\mathbb{R}} (V(t, S_-(1+x)) - V(t, S_-))N(dx, dt). \tag{4.42}$$

The terminal value of a portfolio created by following the self-financing strategy ϑ is a martingale given by

$$\int_0^T \vartheta_t dS_t = \int_0^T \vartheta_t S_{t-} dX_t = \int_0^T \vartheta_t S_{t-}\sigma_t dB_t + \int_0^t \int_{\mathbb{R}} \vartheta_t S_{t-} x N(dt, dx). \tag{4.43}$$

Subtracting (4.43) from (4.42) we get

$$\begin{aligned}\mathbb{E}(I) + L_T = V(0,S_0) &+ \int_0^T \left(\frac{\partial V}{\partial S} - \vartheta\right) S_-\sigma dB \\ &+ \int_0^T \int_{\mathbb{R}} (V(t, S_-(1+x)) - V(t, S_-) - S_-\vartheta x)N(dx, dt).\end{aligned}$$

Because we assume that I is not attainable, we have $I - G_T(\vartheta) = \mathbb{E}(I) + L_T \neq 0$. Thus by Proposition 2.11, we can express $\mathbb{E}(I) + L_T$ as a stochastic exponential, i.e.

$$\mathbb{E}(I) + L_T = \mathcal{E}\left(\int G_u dB_u + \int\int h(u,x)N(du,dx)\right)_T$$

with G and h given by (4.41). Recall the following identity of stochastic exponentials

$$\begin{aligned}\mathcal{E}\left(\int (p-1)G_u dB_u + \int\int g_{p-1}(h(u,x))N(du,dx)\right) = \\ C \operatorname{sgn}\left(\mathcal{E}\left(\int G_u dB_u + \int\int h(u,x)N(du,dx)\right)\right) \\ \times \left|\mathcal{E}\left(\int G_u dB_u + \int\int h(u,x)N(du,dx)\right)\right|^{p-1},\end{aligned} \tag{4.44}$$

where C is the normalizing constant to make right-hand side of (4.44) a martingale and $g_p(h) = \text{sgn}(h+1)|h+1|^p - 1$. Comparing equations (4.39) and (4.44), it follows that the p-optimal hedging strategy ϑ satisfies

$$\mathbb{E}\left[\mathcal{E}\left((p-1)\int G_u dB_u + \int\int g_{p-1}(h(u,x))N(du,dx)\right) G_T(\psi)\right] = 0$$

for any $\psi \in \theta^p$. Thus, the optimal hedging strategy is given as the solution to the following equation

$$(p-1)G_t\sigma + \int_{\mathbb{R}} g_{p-1}(h(t,x))x\nu(dx) = 0, \qquad \text{for } 0 \leq t \leq T.$$

□

When the solution to (4.40) exists, it is unique as the left hand side of (4.40) is monotonous in ϑ, decreasing when $\mathcal{E}\left(\int GdB + \int\int hN\right)_- > 0$ and increasing when $\mathcal{E}\left(\int GdB + \int\int hN\right)_- < 0$. For $p = 2$, we recover the expression for mean-variance hedging strategy in the martingale setting, namely

$$\vartheta = \frac{\sigma^2 \frac{\partial V}{\partial S} + \frac{1}{S}\int_{\mathbb{R}} x(V(t,S_-(1+x)) - V(t,S_-))\nu(dx)}{\sigma^2 + \int_{\mathbb{R}} x^2\nu(dx)}.$$

Example 4.1. *In this example, we use the short notation* $\mathcal{E}_- = \mathcal{E}\left(\int GdB + \int\int hN\right)_-$. *For* $q = 1.5$, *assuming that* $h(t,x) > -1$ *for all* x, *the optimal hedging strategy* ϑ *is given by the solution of the quadratic equation* $a\vartheta^2 + b\vartheta + c = 0$, *where*

$$\begin{aligned}
a &= \frac{S_-^2}{\mathcal{E}_-}\int x^3\nu(dx), \qquad (4.45)\\
b &= -2S_-\left(\sigma^2 + \int (1 + \frac{(V(t,S_-(1+x)) - V(t,S_-))}{\mathcal{E}_-}x^2\nu(dx)\right),\\
c &= 2S_-\sigma^2\frac{\partial V}{\partial S}\\
&\quad + \int\left(\frac{(V(t,S_-(1+x)) - V(t,S_-))^2}{\mathcal{E}_-} + 2(V(t,S_-(1+x) - V(t,S_-))\right)x\nu(dx).
\end{aligned}$$

Let $X_t = (N_1(t) - N_2(t))/2$ *where* N_1 *and* N_2 *are independent Poisson processes of rate 1. Thus* X *has a Lévy measure* $\nu = \delta_{-1/2} + \delta_{1/2}$, *and from the equation* (4.45) *we have* $a = 0$. *The* 3*-optimal hedging strategy is given by*

$$\vartheta = \frac{\sigma^2\frac{\partial V}{\partial S} + \frac{1}{S}\int\left((V(t,S_-(1+x)) - V(t,S_-)) + \frac{(V(t,S_-(1+x)) - V(t,S_-))^2}{2\mathcal{E}_-}\right)x\nu(dx)}{\sigma^2 + \int(1 + \frac{(V(t,S_-(1+x)) - V(t,S_-))}{\mathcal{E}_-}x^2\nu(dx)}. \qquad (4.46)$$

As the assumption $h(t,x) > -1$ for all x involves the solution ϑ given by (4.46), *one needs to check that the assumption is valid. This also means that ϑ is well-defined as denominator in* (4.46) *is non-zero because the assumption implies that ϑ is bounded. When the assumption is not satisfied, the optimal hedge can still be calculated relatively easy, but the resulting formulas will be longer as they need to take into account the changing sign in the function $g_p(h(t,x))$.*

In general, equation (4.40) has to be solved numerically. Recall that the left hand side of (4.40) is monotonous in ϑ. Thus, when there are bounds for ϑ, equation (4.40) can be solved by some root finding method. In the following example we solve the p-optimal hedging problem numerically for European call option in the Merton model.

Example 4.2. *Let us consider Merton jump-diffusion model, cf. Merton (1976), given by*

$$S_t = S_0 \exp(\mu t + \sigma B_t + \sum_{i=1}^{N_t} Y_i).$$

Here N is a Poisson process with intensity λ and independent from the Brownian motion B, and $Y_i \sim N(m, \delta^2)$ are i.i.d. random variables independent from B and N. This is an example of exponential Lévy process. The price of European options in Merton's model can be written as an exponentially converging series of Black-Scholes option prices with different stock prices and volatilities. Let $C^{BS}(\tau, S, \sigma)$ denote the price of European option of a given payoff with time to maturity $\tau = T - t$, current stock price S and volatility σ. By conditioning on the number of jumps N_t, it can be shown that the European price of this option in Merton's model is given by

$$C^M(t,S) = \sum_{n \geq 0} \frac{\exp(-\lambda\tau)(\lambda\tau)^n}{n!} C^{BS}(\tau, S_n, \sigma_n), \tag{4.47}$$

where

$$S_n = S \exp\left(m + \frac{n\delta^2}{2} - \lambda\tau \exp\left(m + \frac{\delta^2}{2}\right) + \lambda\tau\right),$$
$$\sigma_n = \sigma^2 + \frac{n\delta^2}{\tau}.$$

For derivation of the above results, see Merton (1976) or Cont and Tankov (2004). The Merton model's Δ can be found by differentiating equation (4.47)*, which for a European call with strike K leads to*

$$\Delta = \frac{\partial C}{\partial S} = \sum_{n \geq 0} \frac{\exp(-\lambda\tau)(\lambda\tau)^n}{n!} \Phi(d_n),$$

where Φ is the cumulative distribution function of the standard normal random variable and

$$d_n = \frac{\log(S/K) + \sigma^2/2\tau + n\delta^2/2}{\sqrt{\sigma^2\tau + n\delta^2}}.$$

First, we compare the p-optimal hedging strategies at time 0. Note that the p-optimal hedging strategies are path-dependent for $p \neq 2$, which is evident from the denominator in equations given by (4.41)*. At time* 0*, the denominator is equal to 1. In this case, the left hand side of* (4.40) *is monotonically decreasing in ϑ, and we find the root numerically by binary chopping. Both Figure 4.1 and Figure 4.2 depict various optimal hedging strategies for European call option with strike $K = 120$, but with different model parameters. The model parameters have been chosen so that it is clear that there is no ordering among the p-optimal hedging strategies, or between the Merton's Δ and the set of p-optimal hedging strategies. For example, while the Figure 4.1 might suggests that the optimal hedge is increasing function of p, we can see in Figure 4.2 that the mean-variance hedge is below the optimal hedges for $p \neq 2$.*

We now proceed to simulate the paths of S to illustrate how the optimal hedging strategies evolve through time. As we assume that interest rates are zero, S will be a martingale under measure $\mathbb{P}$ if and only if

$$\mu + \frac{\sigma^2}{2} + \mathbb{E}[\exp(Y_i) - 1] = \mu + \frac{\sigma^2}{2} + \lambda \exp\left(m + \frac{\delta^2}{2} - 1\right) = 0. \tag{4.48}$$

The paths of stock price in Merton's model are easy to simulate, for simple algorithms see Cont and Tankov (2004). Figure 4.3 shows the optimal hedging strategies for one sample path finishing in-the-money, while Figure 4.4 shows the optimal hedging strategies for another sample path finishing out-of-the-money. The model parameters for both Figures were as follows: $\sigma = 0.1$, $\lambda = 1$, $m = 0$, $\delta = 0.1$. Note the immediate change in the hedging strategy following the jumps in the underlying. Recall that in the martingale case the mean-variance hedge is equal to the *locally-risk-minimizing hedge. Note that while the Merton's* Δ *hedge converges to* 1 *and* 0 *for in-the-money and out-of-the-money call option, respectively, the p-optimal hedges that take into account the risk from possible jumps in the stock price show no such behaviour as time approaches expiry. Note that in the case of continuous processes, it follows from* (4.40) *that p-optimal hedging strategies are all same and equal to model's* Δ, *and thus they approach* 1 *and* 0 *at expiry. This is no longer the case in the semimartingale setting, where for example the mean-variance hedge is different from model's* Δ *and exhibits the similar behaviour to p-optimal hedging strategies for discontinuous processes in the martingale setting, see Heath, Platen and Schweizer (2001).*

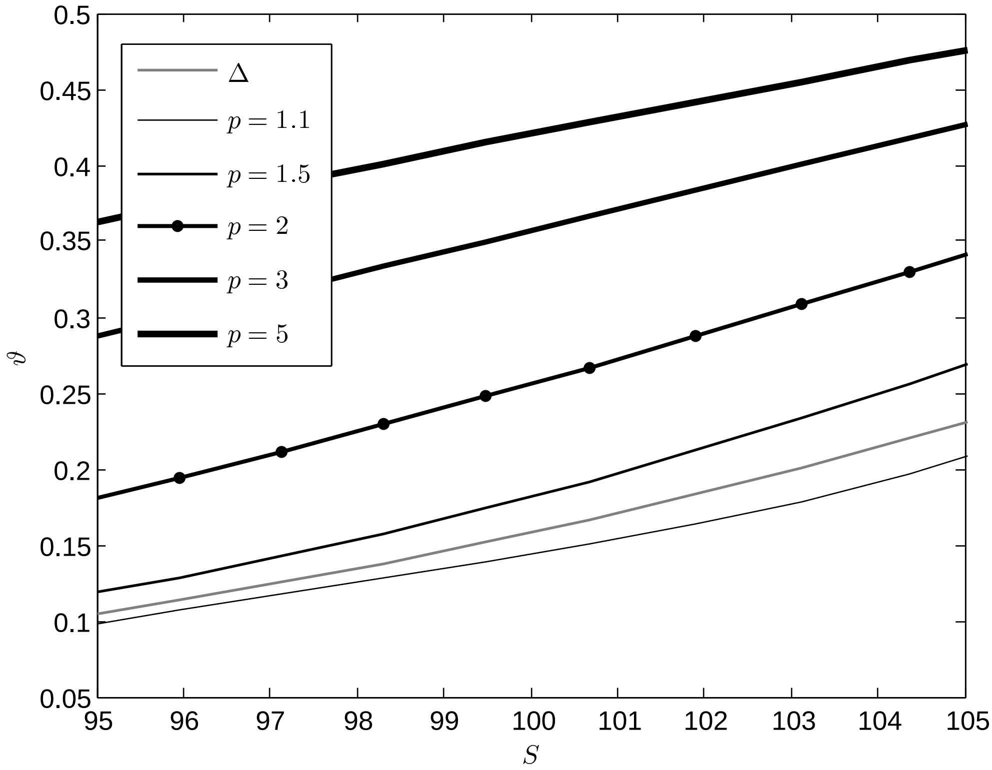

Figure 4.1: Optimal hedging strategies in Merton's model at time 0 *for the call option with strike 120. Model parameters:* $\sigma = 0.1$, $\lambda = 1$, $m = 0$, $\delta = 0.2$.

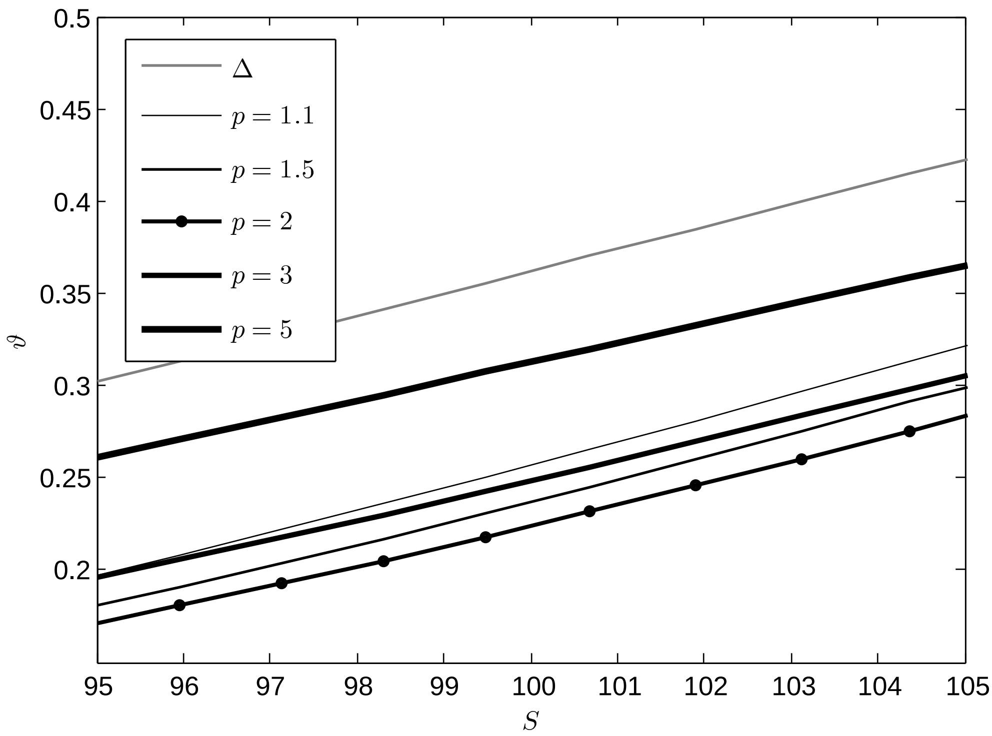

Figure 4.2: Optimal hedging strategies in Merton's model at time 0 *for the call option with strike 120. Model parameters:* $\sigma = 0.2$, $\lambda = 3$, $m = -0.1$, $\delta = 0.2$.

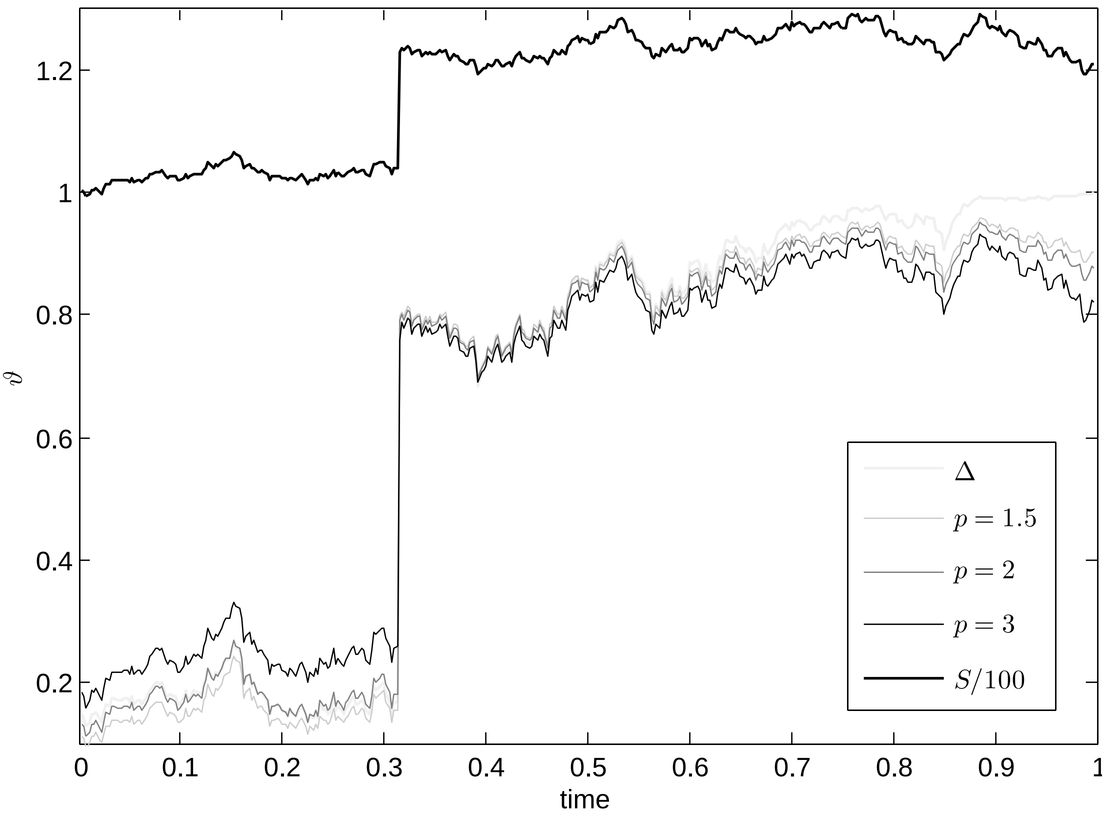

Figure 4.3: Hedging strategies for in-the-money sample path, strike 115.

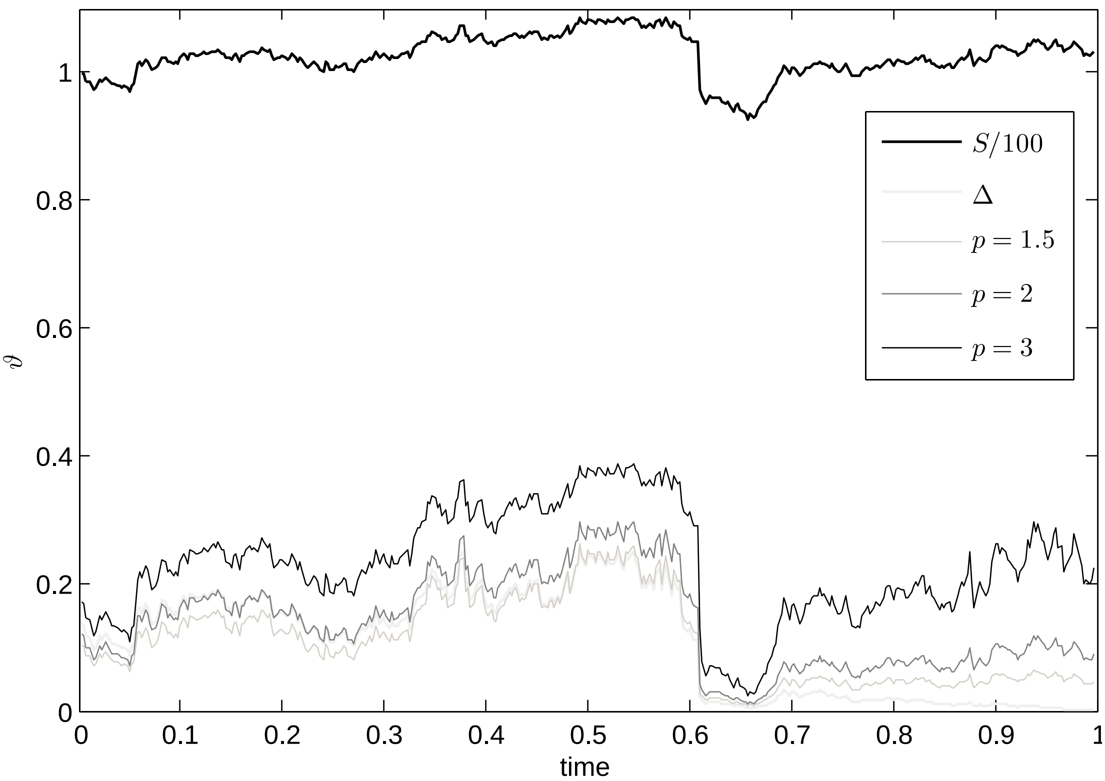

Figure 4.4: Hedging strategies for out-of-the-money sample path, strike 110.

Chapter 5

Numerical results

5.1 Introduction

In this section we specify both the price process and the variance process, and set up the finite difference method to obtain a numerical solution to the partial integro-differential equation, that is needed in order to find the mean variance hedging strategy. By simulating the processes S and V, we then find approximate mean-variance hedge ratios corresponding to the given sample paths.

5.2 Model specifications

Let us first reiterate the general form of the model we consider:

$$\begin{aligned} dS_t &= S_{t-}\big\{b(t,V_{t-})dt + \sigma(t,V_{t-})dB_t + \delta(t,V_{t-})dJ_t\big\} \\ dV_t &= g(t,V_{t-})dt + \gamma(t,V_{t-})dL_t. \end{aligned}$$

Consider now the support of the Lévy measure of the price process S. Without loss of generality, we assume that $\delta(t,V_{t-}) > 0$, a similar analysis can be done for the case when $\delta(t,V_{t-}) < 0$. For the price process to remain positive, the jumps of J have to be bigger than $-1/\delta(t,V_{t-})$. If one requires that the variance optimal martingale measure $\tilde{\mathbb{P}}$ is a probability measure (not only a signed measure), we need $\hat{h}(t,x) > -1$, which guarantees that stochastic exponential in the expression of the density of $\tilde{\mathbb{P}}$ remains positive, cf. equation (3.5) and Lemma 3.9. This implies, assuming $b(t,V_{t-})+\delta(t,V_{t-})\mathbb{E}(J_1)>0$, that the jumps should be bounded from above by

$$\frac{\sigma^2(t,V_{t-})+\delta^2(t,V_{t-})\int x^2\nu(dx)}{b(t,V_{t-})+\delta(t,V_{t-})\mathbb{E}(J_1)}. \tag{5.1}$$

Let $\nu^\Gamma(dx)$ be the Lévy measure of a Gamma(α,β) process, i.e. $\nu^\Gamma(dx)=\alpha\exp(-\beta x)x^{-1}1_{\{x>0\}}dx$. We choose the Lévy measure of J to be $\nu(\Lambda)=\nu^\Gamma(\{x: e^{-x}-1\}\in\Lambda)=\int 1_\Lambda(e^{-x}-1)\nu(dx)$, which means that ν is supported on $(-1,0]$. The previsible part of pure jump Lévy process J is given by

$$\mathbb{E}(J_1)=\int_{0+}^{\infty}(e^{-x}-1)\nu^\Gamma(dx)=\alpha\int_{0+}^{\infty}\frac{e^{-(1+\beta)x}-e^{-\beta x}}{x}dx$$

and predictable quadratic variation of J is given by

$$\langle J\rangle_1=\int_{0+}^{\infty}(e^{-x}-1)^2\nu^\Gamma(dx)=-\alpha\ln\frac{(\beta+1)^2}{\beta^2+2\beta}.$$

Recall the notation and how the change of measure affects jumps processes:

$$\begin{aligned}
N(dt,dx)&=J(dt,dx)-\nu(dx)dt,\\
\tilde{N}(dt,dx)&=J(dt,dx)+(\hat{h}+1)\nu(dx)dt=J(dt,dx)+\hat{\nu}(dt,dx)\\
&=N(dt,dx)-\hat{h}\nu(dx)dt,\\
N^{\mathbb{R}}(dt,dx)&=J(dt,dx)-(\hat{h}+1)\hat{\nu}(dt,dx)=\tilde{N}(dt,dx)-\hat{h}\hat{\nu}(dt,dx),
\end{aligned}$$

which shows that the compensator of J under the measure $\mathbb{R}$ (cf. equation (4.10)) is given by

$$\nu_t^{\mathbb{R}}(dt,dx)=(\hat{h}(t,x)+1)^2\nu(dx)dt.$$

We set $\delta(t,V_{t-}):=1$. As a result, the price process is guaranteed to stay positive as the jumps are bigger than $-1/\delta(t,V_{t-})=-1$. Also, the variance optimal martingale measure is a probability measure because the condition (5.1) for the jump measure is satisfied. Determining the dynamics of V under $\tilde{\mathbb{P}}$ requires to evaluate $R(t,V_t)$ from equation (4.14). We set up the model parameters so that the drift of the volatility process under $\tilde{\mathbb{P}}$ is related to the conditional Laplace transform of the integrated variance process. Setting

$$\sigma(t,V_{t-})=\sqrt{V_{t-}},\qquad b(t,V_{t-})=-\mathbb{E}(J_1)+\varsigma\sqrt{V_{t-}^2+V_{t-}\int x^2\nu(dx)}, \tag{5.2}$$

implies that the mean-variance tradeoff process $\hat{K}_t = \varsigma^2 \int_0^t V_u du$ is random and unbounded and

$$R(t, V_t) = \mathbb{E}\left[\exp\left(-\varsigma^2 \int_t^T V_u du\right) \middle| \mathcal{F}_t\right]. \tag{5.3}$$

From the assumption $b(t, V_{t-}) + \delta(t, V_{t-})\mathbb{E}(J_1) > 0$ we have that $\varsigma > 0$. The right hand side of (5.3) can be calculated explicitly for several models of volatility processes. In such models, we can then find the drift of the volatility process under both measures $\tilde{\mathbb{P}}$ and $\mathbb{R}$. We consider an Gamma-Ornstein-Uhlenbeck process that takes the following form

$$dV_t = -\rho V_t dt + dZ_{\rho t}. \tag{5.4}$$

Here Z is a compound Poisson process with exponential jump size distribution with parameter β, and ρ is a constant. The stationary measure of V is Gamma(α,β). The Laplace transform of the integrated variance process has the explicit form

$$\mathbb{E}\left[\exp\left(z \int_t^T V_u du\right) \middle| \mathcal{F}_t\right] = \exp\left\{z\epsilon(t,T)V_t + \int_t^T \rho \frac{\alpha z \epsilon(u,T)}{\beta - z\epsilon(u,T)} du\right\}, \tag{5.5}$$

where $\epsilon(t, T)$ denotes the deterministic function $\epsilon(t, T) = (1 - e^{-\rho(T-t)})/\rho$, see Nicolato and Venardos (2003). Equations (4.15) and (4.16) now simplify to

$$\begin{aligned} F_t &= \frac{\gamma(t, V_{t-}) \frac{\partial R}{\partial v}(t, V_{t-})}{R(t, V_{t-})} = -\varsigma^2 \epsilon(t,T)\gamma(t, V_{t-}), \\ f(t, x) &= \frac{\Delta R(t, V_t)}{R(t, V_{t-})} = \exp(-\varsigma^2 \epsilon(t,T)\gamma(t, V_{t-})x) - 1. \end{aligned}$$

The drift of the compound Poisson process Z is given by α/β. From equation (5.4), $\gamma(t, V_{t-}) = 1$ and $g(t, V_{t-}) = -\rho V_{t-}$. The drift of the variance process under $\tilde{\mathbb{P}}$ is given by

$$\tilde{g}(t, V_{t-}) = -\rho\left\{V_{t-} - \frac{\alpha\beta}{(\beta + \varsigma^2\epsilon(t,T))^2}\right\},$$

and the $\mathbb{R}$-compensator of V is given by

$$\nu_t^{V,\mathbb{R}}(x) = \alpha\beta \exp\big(-(\beta + \varsigma^2\epsilon(t,T))x\big).$$

5.3 European put

5.3.1 Change of variables

The Markovian structure of the model implies that finding a Galtchouk-Kunita-Watanabe decomposition of $I/\tilde{Z}_T$ under $\mathbb{R}$ is equivalent to finding the solution to a PIDE problem. Recall definition of the $\mathbb{R}^2$-valued process Y, given by

$$Y^0 := \tilde{Z}^{-1}, \quad Y^1 := S\tilde{Z}^{-1}.$$

As the goal in this chapter is to derive the optimal hedging strategies, we now introduce a filtration generated by the processes S and V, and denote it by $\mathcal{F}^{(S,V)}$. The filtration $\mathcal{F}^{(S,V)}$ is the same as the one generated by the processes Y^0,Y^1 and V, and it is more appropriate to use for the task at hand compared to the, in general richer, filtration $\mathcal{F}$ generated by driving processes B, J and L. Let us define

$$u(t, Y_t^0, Y_t^1, V_t) = \mathbb{E}_{\mathbb{R}}[I/\tilde{Z}_T|\mathcal{F}_t^{(S,V)}] = \mathbb{E}_{\mathbb{R}}\left[\upsilon(Y_T^0, Y_T^1, V_T)|\mathcal{F}_t^{(S,V)}\right].$$

Assuming regularity of the model parameters, the Feynman-Kac representation theorem implies

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \tilde{g}(t,v)\frac{\partial u}{\partial v} + \frac{1}{2}\left(\gamma^2\frac{\partial^2 u}{\partial v^2} + (y\sigma(1+\hat{\lambda}S))^2\frac{\partial^2 u}{\partial y^2} + (x\sigma\hat{\lambda}S)^2\frac{\partial^2 u}{\partial x^2}\right)\\
&- xy\sigma^2\hat{\lambda}S(1+\hat{\lambda}S)\frac{\partial^2 u}{\partial x\partial y} + \int\left\{u(t,x,y,v+\gamma z) - u(t,x,y,v) - \gamma z\frac{\partial u}{\partial v}\right\}\nu_t^{V,\mathbb{R}}(dz)\\
&+\int\left\{u\left(t, \frac{x}{1-\delta z\hat{\lambda}S}, \frac{y+\delta yz}{1-\delta z\hat{\lambda}S}, v\right) - u(t,x,y,v)\right.\\
&\qquad\left. - \frac{\delta x\hat{\lambda}Sz}{1-\delta z\hat{\lambda}S}\frac{\partial u}{\partial x} - \frac{\delta z(1+\hat{\lambda}S)y}{1-\delta z\hat{\lambda}S}\frac{\partial u}{\partial y}\right\}\nu_t^{\mathbb{R}}(dz) = 0,
\end{aligned} \tag{5.6}$$

with terminal condition

$$u(T,x,y,v) = w(x,y,v). \tag{5.7}$$

For the contingent claim we consider a put option with maturity T and strike K. Thus $I = (K - S_T)^+$ and $w(Y_T^0, Y_T^1, V_T) = (KY_T^0 - Y_T^1)^+$. Apart from the terminal condition, the boundary conditions have to be defined as well. For the moment, consider the standard pricing problem that is set under some martingale measure $\mathbb{Q}$

and the function $u_{\mathbb{Q}}$ is defined as $u_{\mathbb{Q}}(t, S_{t-}, V_t) = \mathbb{E}_{\mathbb{Q}}[I|\mathcal{F}_t]$. The boundary conditions for a put option can be defined as (Ikonen and Toivanen (2004)):

$$
\begin{aligned}
u_{\mathbb{Q}}(t, 0, V_t) &= K, & u_{\mathbb{Q}}(t, S_{t-}, 0) &= (K - S_{t-})^+, \\
\frac{\partial u_{\mathbb{Q}}}{\partial S_t}(t, \infty, V_t) &= 0, & \frac{\partial u_{\mathbb{Q}}}{\partial V_t}(t, S_{t-}, \infty) &= 0.
\end{aligned}
$$

Our aim is to transform these conditions into our problem. However, because the problem is now three dimensional (in the space), two additional boundary conditions are needed, and because the payoff is linear in Y^0, we set

$$
\frac{\partial^2 u}{\partial x^2}(t, 0, y, v) = 0, \quad \frac{\partial^2 u}{\partial x^2}(t, \infty, y, v) = 0. \tag{5.8}
$$

Note that $S = Y^1/Y^0$, and consider the cases where S is constant, where S is going to zero and to infinity. Specifying the boundary condition for Y^1 would not be easy and it would probably make more sense to consider other than rectangular region for the solution space, which would complicate the implementation of the finite difference method. Another problem in the implementation of finite difference method is a cross-term due to correlation between Y^0 and Y^1. When using operator splitting method, Yanenko (1971) suggests to treat the correlation term explicitly, for implementation see for example Duffy (2006), however this term was causing spurious oscillations of the solution in our tests. To address both issues, the change of variables technique can be used to transform the problem in a way that simplifies specification of the boundary conditions and at the same time removes the cross term. To this end, define

$$
\alpha = \frac{\ln Y^1}{1 + \hat{\lambda} S} - \frac{\ln Y^0}{\hat{\lambda} S}, \quad \beta = \ln(Y^1/Y^0),
$$
$$
\omega(t, \alpha, \beta, v) = u(t, \exp(\hat{\lambda} S(\beta - (1 + \hat{\lambda} S)\alpha)), \exp((1 + \hat{\lambda} S)(\beta - \hat{\lambda} S \alpha)), v).
$$

Using the change of variables formula, the original PIDE can be rewritten to the following (all partial derivatives evaluated at (t, α, β, v)):

$$\begin{aligned}
&\frac{\partial \omega}{\partial t} + \tilde{g}\frac{\partial \omega}{\partial v} + \frac{\gamma^2}{2}\frac{\partial^2 \omega}{\partial v^2} - \frac{\sigma^2}{2}\frac{\partial \omega}{\partial \alpha} + \sigma^2 \left(\frac{(\hat{\lambda}S)^2}{2} - \hat{\lambda}S(1+\hat{\lambda}S) + \frac{(1+\hat{\lambda}S)^2}{2} \right) \frac{\partial^2 \omega}{\partial \beta^2} \\
&+ \sigma^2 \frac{(\hat{\lambda}S)^2 - (1+\hat{\lambda}S)^2}{2} \frac{\partial \omega}{\partial \beta} \\
&+ \int \left\{ \omega\left(t, \alpha + \frac{\ln(1+\delta z)}{1+\hat{\lambda}S} - \frac{\ln(1-\delta z\hat{\lambda}S)}{1+\hat{\lambda}S} + \frac{\ln(1-\delta z\hat{\lambda}S)}{\hat{\lambda}S}, \beta + \ln(1+\delta z), v \right) \right. \\
&\qquad \left. -\omega(t,\alpha,\beta,v) - \frac{\delta x}{1-\delta x\hat{\lambda}S}\frac{\partial \omega}{\partial \beta} \right\} \nu_t^{\mathbb{R}}(dz) \\
&+ \int \left(\omega(t,\alpha,\beta,v+\gamma z) - \omega(t,\alpha,\beta,v) - \gamma z \frac{\partial \omega}{\partial v} \right) \nu_t^{V,\mathbb{R}}(dz) = 0, \qquad (5.9)
\end{aligned}$$

with terminal condition

$$\omega(T,\alpha,\beta,v) = (K\exp(\hat{\lambda}S(\beta - (1+\hat{\lambda}S)\alpha)) - \exp((1+\hat{\lambda}S)(\beta - \hat{\lambda}S\alpha)))^+,$$

and a set of boundary conditions

$$\begin{aligned}
\omega(t,\alpha,-\infty,v) &= Ke^{\hat{\lambda}S(\beta-(1+\hat{\lambda}S)\alpha)}, & \frac{\partial \omega}{\partial \beta}(t,\alpha,\infty,v) &= 0, \\
\frac{\partial \omega}{\partial \alpha}(t,-\infty,\beta,v) &= 0, & \frac{\partial \omega}{\partial \alpha}(t,\infty,\beta,v) &= 0, \\
\omega(t,\alpha,\beta,0) &= (Ke^{\hat{\lambda}S(\beta-(1+\hat{\lambda}S)\alpha)} - e^{(1+\hat{\lambda}S)(\beta-\hat{\lambda}S\alpha)})^+, & \frac{\partial \omega}{\partial v}(t,\alpha,\beta,\infty) &= 0.
\end{aligned}$$

5.3.2 Finite difference method

We are going to solve the PIDE given by (5.9) by finite difference method. We use the explicit-implicit time stepping scheme used in Cont and Voltchkova (2005a) and the general operator splitting method to deal with dimensionality problem, see for example Duffy (2006). Let us define the following integro-differential operators:

$$\begin{aligned}
\mathfrak{L}_t^{\alpha}\omega &= -\frac{\sigma^2}{2}\frac{\partial \omega}{\partial \alpha}, \\
\mathfrak{L}_t^{\beta}\omega &= \sigma^2 \left(\frac{(\hat{\lambda}S)^2}{2} - \hat{\lambda}S(1+\hat{\lambda}S) + \frac{(1+\hat{\lambda}S)^2}{2} \right) \frac{\partial^2 \omega}{\partial \beta^2} + \sigma^2 \frac{(\hat{\lambda}S)^2 - (1+\hat{\lambda}S)^2}{2}\frac{\partial \omega}{\partial \beta}, \\
\mathfrak{L}_t^{V}\omega &= \tilde{g}(t,v)\frac{\partial \omega}{\partial v} + \frac{\gamma^2}{2}\frac{\partial^2 \omega}{\partial v^2} + \int \left(\omega(t,\alpha,\beta,v+\gamma z) - \omega(t,\alpha,\beta,v) - \gamma z\frac{\partial \omega}{\partial v} \right) \nu_t^{V,\mathbb{R}}(dz).
\end{aligned}$$

Using the knowledge about the variance process modelled by Gamma-OU process, namely having no second order term and the fact that $\nu_t^{V,\mathbb{R}}(\mathbb{R}) < \infty$, we have

$$
\begin{aligned}
\mathfrak{L}_t^V \omega = &\left(\tilde{g}(t,v) - \int \gamma z \nu_t^{V,\mathbb{R}}(dz)\right) \frac{\partial \omega}{\partial v} \\
&+ \int \omega(t,\alpha,\beta,v+\gamma z)\nu_t^{V,\mathbb{R}}(dz) - \omega(t,\alpha,\beta,v)\nu_t^{V,\mathbb{R}}(\mathbb{R}).
\end{aligned}
$$

Similar technique can be applied to the integral with respect to $\nu_t^{\mathbb{R}}(dz)$ in (5.9), after using the approximation of infinity activity Lévy process by a compound Poisson process (in our case we approximate the jumps smaller than ϵ by their expectation). Defining the following integral operator

$$
\begin{aligned}
&\mathfrak{L}_t^\epsilon \omega = -\omega(t,\alpha,\beta,v)\nu_t^{\mathbb{R}}\big(\mathbb{R}\backslash(-\epsilon,\epsilon)\big) \\
&\int_{|z|>\epsilon} \omega\left(t, \alpha + \frac{\ln(1+\delta z)}{1+\hat{\lambda}S} - \frac{\ln(1-\delta z\hat{\lambda}S)}{1+\hat{\lambda}S} + \frac{\ln(1-\delta z\hat{\lambda}S)}{\hat{\lambda}S}, \beta + \ln(1+\delta z), v\right) \nu_t^{\mathbb{R}}(dz),
\end{aligned}
$$

the suggested splitting scheme for the numerical approximation of the solution to (5.9) is then given by the following set of equations:

$$
\begin{aligned}
\frac{\partial \omega}{\partial t} &= \mathfrak{L}_t^\alpha \omega + \frac{1}{2}\mathfrak{L}_t^\epsilon \omega, \\
\frac{\partial \omega}{\partial t} &= \mathfrak{L}_t^\beta \omega + \frac{1}{2}\mathfrak{L}_t^\epsilon \omega - \left(\int_{|z|>\epsilon} \frac{\delta x}{1-\delta x\hat{\lambda}S}\nu_t^{\mathbb{R}}(dz)\right)\frac{\partial \omega}{\partial \beta}, \\
\frac{\partial \omega}{\partial t} &= \mathfrak{L}_t^V \omega.
\end{aligned}
$$

The explicit-implicit time stepping scheme is used for all three legs, differential part dealt with implicitly and integral part dealt with explicitly, cf. Cont and Voltchkova (2005a).

5.3.3 Computation of hedge ratios

Once the solution to the PIDE given by (5.9) is found, we can find approximate hedge ratios by simulating the vector (S,V) in the interval $[0,T]$. The first order Euler-type approximation scheme is used to obtain the sample path $(\bar{S}_{t_i}, \bar{V}_{t_i})$, for $i \in \{1,N\}$ and $t_i = iT/N$. The partial derivatives appearing in the integrand of the Galtchouk-Kunita-Watanabe decomposition given by (4.31) can be approximated by finite differences. Due to the fact that the simulated processes S and V might not lie on the grid at which the PIDE was solved, the three-dimensional interpolation method is used to obtain the finite differences for $\partial\omega/\partial\alpha$ and $\partial\omega/\partial\beta$, and then appealing to the change of variables formula, we have

$$\begin{aligned}\frac{\partial u}{\partial x} &= -\frac{1}{\hat{\lambda}Sx}\frac{\partial\omega}{\partial\alpha} - \frac{1}{x}\frac{\partial\omega}{\partial\beta},\\ \frac{\partial u}{\partial y} &= \frac{1}{(1+\hat{\lambda}S)y}\frac{\partial\omega}{\partial\alpha} + \frac{1}{y}\frac{\partial\omega}{\partial\beta}.\end{aligned}$$

The approximate hedge ratios $\bar{\bar{\vartheta}}$ are found by using the Euler approximation scheme:

$$\bar{\bar{\vartheta}}_{t_i} = \hat{\lambda}_{t_i}\psi^*_{t_i} + \zeta_{t_i}\left(\sum_{j=1}^{i}\psi^*_{t_{j-1}}(\bar{Y}^0_{t_j} - \bar{Y}^0_{t_{j-1}}) + u(0, Y^0_0, Y^1_0, V_0)\right), \qquad i \in \{0, N\}.$$

One sample path of the vector $(\bar{S}, \bar{V})$ with the corresponding mean variance hedging strategy can be seen on Figure (5.5). The strike of the put option was set to 100, $S_0 = 100$, $V_0 = 0.1$, and the computational grid for (T, Y^0, Y^1, V) of the size $100 \times 100 \times 100 \times 50$. We can see in (5.5) that the mean-variance hedge at the expiry is not approaching -1 when the option is finishing in the money, as would be the case in the Black-Scholes framework.

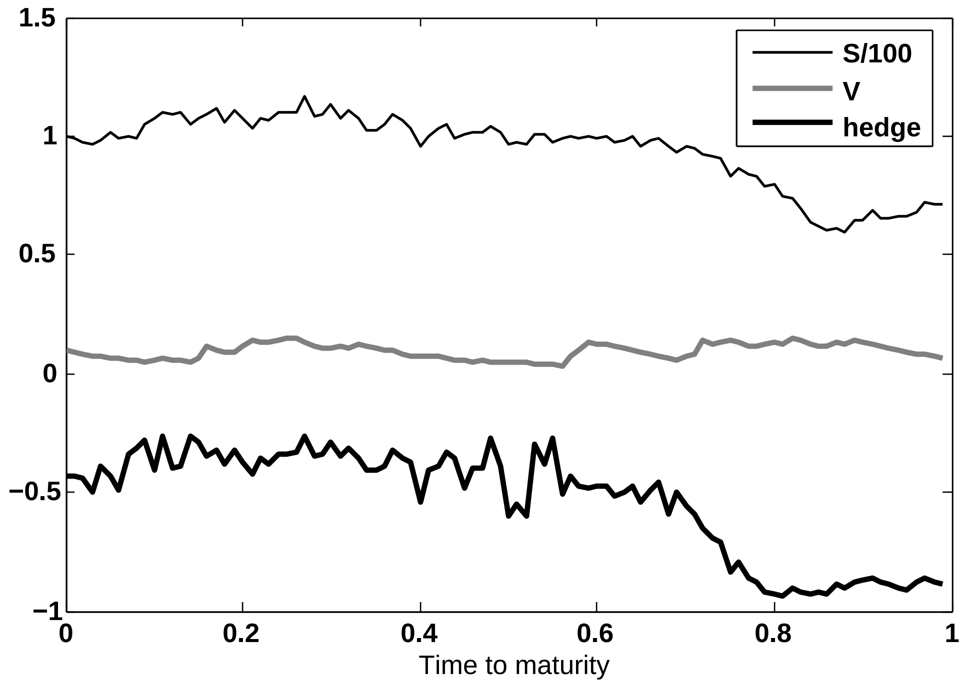

Figure 5.5: The mean-variance hedging strategy

5.4 Option on realized variance

In this section, we consider hedging of the put option on realized variance (with notional 1), which has the following payoff function

$$I = \left(K - \frac{1}{T} \int_0^T \sigma_t^2 dt \right)^+ . \tag{5.10}$$

This has the advantage that under $\mathbb{R}$, Y^1 does not come up in the resulting payoff function $I/\tilde{Z}_T$. However, from the analysis of Asian option pricing, we know that to preserve the Markov property we need to introduce another state variable, and thus after applying the Feynman-Kac formula, the resulting PIDE will again be three-dimensional. The integrated variance process, $I(t) = \int_0^t V_u du$, is the newly introduced variable so that $I/\tilde{Z}_T = Y_T^0 (K - I_T/T)^+$, and the pricing/hedging PIDE is stated in the following

Theorem 5.1. *Let $\omega(t,x,z,v) = \mathbb{E}_{\mathbb{R}}[I/\tilde{Z}_T | \ln Y_t^0 = \alpha, I_t = \beta, V_t = v]$. Then the following PIDE holds*

$$\begin{aligned} &\frac{\partial \omega}{\partial t} + \tilde{g}(t,v)\frac{\partial \omega}{\partial v} + \frac{\gamma^2}{2}\frac{\partial^2 \omega}{\partial v^2} + \frac{(\sigma\hat{\lambda}S)^2}{2}\left(\frac{\partial^2 \omega}{\partial \alpha^2} - \frac{\partial \omega}{\partial \alpha}\right) \\ &+ \int \left(\omega(t, \alpha - \ln(1 - \delta z \hat{\lambda} S), \beta, v) - \omega(t,\alpha,\beta,v) - \frac{\delta x \hat{\lambda} S}{1 - \delta x \hat{\lambda} S}\frac{\partial \omega}{\partial \alpha}\right) \nu_t^{\mathbb{R}}(dz) \\ &+ \int \left(\omega(t,\alpha,\beta,v+\gamma z) - \omega(t,\alpha,\beta,v) - \gamma z \frac{\partial \omega}{\partial v}\right) \nu_t^{V,\mathbb{R}}(dz) = 0, \end{aligned} \tag{5.11}$$

The PIDE itself seems easier than in the case of European put option, but the problem is in the specification of the boundary conditions. This is a standard problem when pricing/hedging Asian option. One powerful approach was developed by Vecer (2001), by applying the change of numéraire technique. This decreases dimensionality of the PIDE and removes the boundary condition problem at the same time. While the change of numéraire technique can be used without problem here, this approach won't lead to any simplifications due to dependence of the process Y^0 on the variance, which implies that the reduced model would not be a Markov process.

Bibliography

Applebaum, D. (2004): *Lévy processes and stochastic calculus.* Cambridge studies in advanced mathematics.

Arai, T. (2001): The p-optimal martingale measure in continuous trading models. *Statistics & Probability Letters* 54, 93–99.

Arai, T. (2002): Mean-variance hedging for discontinuous semimartingales. *Tokyo J. Math.* 25, 435–452.

Arai, T. (2004): Minimal martingale measures for jump diffusion processes. *Journal of Applied Probability* 41, 263–270.

Arai, T. (2005a): An extension of mean-variance hedging to the discontinuous case. *Finance and Stochastics* 9, 129–139.

Arai, T. (2005b): Some properties of the variance-optimal martingale measure for discontinuous semimartingales. *Statistics & Probability Letters* 74, 163–170.

Arai, T. (2005c): Some remarks on mean-variance hedging for discontinuous asset price processes. *International Journal of Theoretical and Applied Finance* 8, 425–443.

Arai, T. (2006): $\mathcal{L}^p$-projections of random variables and its applications to finance. Keio Economics Society Discussion Paper Series 06-2.

Bardhan, I. and X. Chao (1995): Martingale analysis for assets with discontinuous returns. *Math. Oper. Res.* 20, 243–256.

Bardhan, I. and X. Chao (1996): On martingale measures when asset returns have unpredictable jumps. *Stochastic Processes and their Applications* 63, 35–54.

Barndorff-Nielsen, O. E., E. Nicolato, and N. Shephard (2002): Some recent developments in stochastic volatility modelling. *Quantitative Finance* 2, 11–23.

Bartle, R. G. (1995): *The elements of integration and Lebesgue measure*. Wiley Classics Library, John Wiley & Sons Inc., New York.

Bender, C. and C. R. Niethammer (2007): On q-optimal signed martingale measures in exponential lévy models. Unpublished.

Biagini, F. (1999): *Quadratic Hedging Approaches For Interest Rate Models With Stochastic Volatility*. Ph.D. thesis, Scuola Normale Superiore, Pisa.

Biagini, F. (2002): Mean-variance hedging for interest rate models with stochastic volatility. *Decisions in Economics and Finance. A Journal of Applied Mathematics* 25, 1–17.

Biagini, F. and P. Guasoni (2002): Mean-variance hedging with random volatility jumps. *Stochastic Analysis and Applications* 20, 471–494.

Biagini, F., P. Guasoni, and M. Pratelli (2000): Mean-variance hedging for stochastic volatility models. *Mathematical Finance* 10, 109–123.

Carr, P., H. Geman, and D. B. Madan (2000): Pricing and hedging in incomplete markets. *Journal of financial economics* 131–167.

Carr, P., H. Geman, D. B. Madan, and M. Yor (2002): The fine structure of asset returns: An empirical investigation. *Journal of Business, Vol. 75, No. 2, April 2002* 75, 305–332.

Carr, P., H. Geman, D. B. Madan, and M. Yor (2003): Stochastic volatility for Lévy processes. *Mathematical Finance* 13, 345–382.

Carr, P. and L. Wu (2004): Time-Changed Lévy Processes and Option Pricing. *Journal of Financial Economics* 17, 113–141.

Černý, A. and J. Kallsen (2008): Mean-variance hedging and optimal investment in Heston's model with correlation. *Math. Finance* 18, 473–492.

Chan, T. (1999): Pricing contingent claims on stocks driven by Lévy processes. *The Annals of Applied Probability* 9, 504–528.

Cherny A.S., S. A. (2002): Change of time and measure for Lévy processes.

Choulli, T., L. Krawczyk, and C. Stricker (1998): $\mathcal{E}$-martingales and their applications in mathematical finance. *Annals of Probability* 26, 876–853.

Choulli, T. and C. Stricker (2005): Minimal Entropy-Hellinger Martingale Measure In Incomplete Markets. *Mathematical Finance* 15, 465–490.

Cont, R. and P. Tankov (2004): *Financial modelling with jump processes*. Chapman & Hall.

Cont, R., P. Tankov, and E. Voltchkova (2007): Hedging with options in models with jumps. In *Stochastic analysis and applications*, vol. 2 of *Abel Symp.* Springer, Berlin, 197–217.

Cont, R. and E. Voltchkova (2005a): A finite difference scheme for option pricing in jump diffusion and exponential Lévy models. *SIAM Journal on Numerical Analysis* 43, 1596–1626.

Cont, R. and E. Voltchkova (2005b): Integro-differential equations for option prices in exponential Lévy models. *Finance and Stochastics* 9, 299–325.

Cox, A. M. G. and D. G. Hobson (2005): Local martingales, bubbles and option prices. *Finance and Stochastics* 9, 477–492.

Delbaen, F., P. Monat, W. Schachermayer, M. Schweizer, and C. Stricke (1997): Weighted norm inequalities and hedging in incomplete markets. *Finance and Stochastics* 1, 181–227.

Delbaen, F. and W. Schachermayer (1996a): Attainable claims with pth moments. *Ann. Inst. H. Poincaré Probab. Statist.* 32, 743–763.

Delbaen, F. and W. Schachermayer (1996b): Corrections: “The variance-optimal martingale measure for continuous processes” [Bernoulli 2 (1996), no. 1, 81–105; MR1394053 (97c:60115)]. *Bernoulli* 2, 379–380.

Delbaen, F. and W. Schachermayer (1996c): The variance-optimal martingale measure for continuous processes. *Bernoulli* 2, 81–105.

Dellacherie, C. and P.-A. Meyer (1982): *Probabilities and potential. B*, vol. 72 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam.

Di Mazi, G. B., Y. M. Kabanov, and V. I. Runggaldier (1994): Mean-square hedging of options on a stock with Markov volatilities. *Rossiĭskaya Akademiya Nauk. Teoriya Veroyatnosteĭ i ee Primeneniya* 39, 211–222.

Duffie, D., D. Filipovič, and W. Schachermayer (2003): Affine processes and applications in finance. *The Annals of Applied Probability* 13, 984–1053.

Duffie, D. and H. R. Richardson (1991): Mean-variance hedging in continuous time. *The Annals of Applied Probability* 1, 1–15.

Duffy, D. (2006): *Finite Difference Methods in Financial Engineering*. Wiley.

Černý, A. and J. Kallsen (2007): On the structure of general mean-variance hedging strategies. *Annals of Probability* 35, 1479–1531.

Černý, A. and J. Kallsen (2008): A counterexample concerning the variance-optimal martingale measure. *Mathematical Finance* 18, 305–316.

Esche, F. (2004): *Two Esseys on Incomplete Markets*. Ph.D. thesis, Technischen Universität Berlin.

Esche, F. and M. Schweizer (2005): Minimal entropy preserves the Lévy property: how and why. *Stochastic Processes and their Applications* 115, 299–327.

Filipovič, D. (2005): Time-inhomogeneous affine processes. *Stochastic Processes and their Applications* 115, 639–659.

Föllmer, H. and M. Schweizer (1991): Hedging of contingent claims under incomplete information. In *Applied Stochastic Analysis* (M. H. A. Davis and R. J. Elliott, eds.). Gordon and Breach, London/New York, 389–414.

Föllmer, H. and D. Sondermann (1986): Hedging of nonredundant contingent claims. In *Contributions to Mathematical Economics: Essays in Honour of G. Debreu* (W. Hilderbrand and A. Mascolell, eds.). Elsevier Science Publishers B.V. (North Holland), Amsterdam, 205–223.

Frittelli, M. (2000): The minimal entropy martingale measure and the valuation problem in incomplete markets. *Mathematical Finance* 10, 39–52.

Fujiwara, T. and Y. Miyahara (2003): The minimal entropy martingale measures for geometric Lévy processes. *Finance and Stochastics* 7, 509–531.

Geman, H., N. El Karoui, and J.-C. Rochet (1995): Changes of numéraire, changes of probability measure and option pricing. *Journal of Applied Probability* 32, 443–458.

Goll, T. and J. Kallsen (2000): Optimal portfolios for logarithmic utility. *Stochastic Process. Appl.* 89, 31–48.

Gourieroux, C., J. P. Laurent, and H. Pham (1998): Mean-variance hedging and numéraire. *Mathematical Finance* 8, 179–200.

Grandits, P. (1999): The p-optimal martingale measure and its asymptotic relation with the minimal-entropy martingale measure. *Bernoulli. Official Journal of the Bernoulli Society for Mathematical Statistics and Probability* 5, 225–247.

Grandits, P. and L. Krawczyk (1998): Closedness of some spaces of stochastic integrals. In *Séminaire de Probabilités, XXXII*, vol. 1686 of *Lecture Notes in Math.* Springer, Berlin, 73–85.

Grandits, P. and T. Rheinländer (2002): On the minimal entropy martingale measure. *The Annals of Probability* 30, 1003–1038.

Gut, A. (2005): *Probability: a graduate course.* Springer Texts in Statistics, Springer, New York.

Heath, D., E. Platen, and M. Schweizer (2001): A comparison of two quadratic approaches to hedging in incomplete markets. *Mathematical Finance* 11, 385–413.

Henderson, V. (2005): Analytical comparisons of option prices in stochastic volatility models. *Mathematical Finance* 15, 49–59.

Heston, S. (1993): A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *Review of Financial Studies* 6, 327–343.

Hipp, C. (1993): Hedging general claims. In *Proceedings of the 3rd AFIR Colloquium Rome*, vol. 2. 603—-613.

Hobson, D. (2004): Stochastic volatility models, correlation, and the q-optimal measure. *Mathematical Finance* 14, 537–556.

Hubalek, F., J. Kallsen, and L. Krawczyk (2006): Variance-optimal hedging for processes with stationary independent increments. *The Annals of Applied Probability* 16, 853–885.

Hubalek, F. and C. Sgarra (2004): Equivalent martingale measures for stochastic volatility models driven by Lévy processes. Unpublished.

Hubalek, F. and C. Sgarra (2006): Esscher transforms and the minimal entropy martingale measure for exponential Lévy models. *Quant. Finance* 6, 125–145.

Hubalek, F. and C. Sgarra (2007): Quadratic hedging for the Bates model. *International Journal of Theoretical and Applied Finance* 10, 873–885.

Ikonen, S. and J. Toivanen (2004): Operator splitting methods for american options with stochastic volatility. In *European Congress on Computational Methods in Applied Sciences and Engineering*.

Jacod, J. (1979): *Calcul stochastique et problèmes de martingales*, vol. 714 of *Lecture Notes in Mathematics*. Springer, Berlin.

Jacod, J. and A. N. Shiryaev (2002): *Limit Theorems for Stochastic Processes.* 2nd ed. Springer Verlag.

Jeanblanc, M., S. Klöppel, and Y. Miyahara (2007): Minimal f^q-martingale measures of exponential Lévy processes. *Ann. Appl. Probab.* 17, 1615–1638.

Kallsen, J. (1998): *Semimartingales Modelling in Finance.* Ph.D. thesis, Albert-Ludwigs-Universität Freiburg.

Kallsen, J. (2000): Optimal portfolios for exponential Lévy processes. *Mathematical Methods of Operations Research* 51, 357–374.

Kallsen, J. (2006): Quadratic hedging in affine stochastic volatility models. Presentation in Pittsburgh.

Kallsen, J. and A. N. Shiryaev (2002): The cumulant process and Esscher's change of measure. *Finance and Stochastics* 6, 397–428.

Kunita, H. (2004): Representation of martingales with jumps and applications to mathematical finance. *Advanced studies in pure mathematics* 41, 209–232.

Kunita, H. and S. Watanabe (1967): On square integrable martingales. *Nagoya Math. J.* 30, 209–245.

Laurent, J. P. and H. Pham (1999): Dynamic programming and mean-variance hedging. *Finance and Stochastics* 3, 83–110.

Li, H., M. T. Wells, and C. L. Yu (2006): A Bayesian Analysis of Return Dynamics with Levy Jumps. *Rev. Financ. Stud.* doi:10.1093/rfs/hhl036.

Lim, A. E. B. (2005): Mean-variance hedging when there are jumps. *SIAM J. Control Optim.* 44, 1893–1922.

Luenberger, D. (1969): *Optimization by Vector Space Methods.* Wiley-Interscience.

Merton, R. C. (1976): Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* 3, 125–144.

Miyahara, Y. (2004): A note on Escher transformed martingale measures for geometric Lévy processes. Unpublished.

Moller, T. (2000): *Quadratic hedging approaches and indifference pricing in insurance*. Ph.D. thesis, University of Copenhagen.

Monat, P. and C. Stricker (1995): Föllmer-Schweizer decomposition and mean-variance hedging for general claims. *Ann. Probab.* 23, 605–628.

Nicolato, E. and E. Venardos (2003): Option pricing in stochastic volatility models of the Ornstein-Uhlenbeck type. *Mathematical Finance* 13, 445–466.

Overberg, L. (1995): On the predictable representation property for super-processes. *Séminaire de probabilités de Strasbourg* 29, 108–116.

Pham, H. (2000): On quadratic hedging in continuous time. *Mathematical Methods of Operations Research* 51, 315–339.

Pham, H., T. Rheinländer, and M. Schweizer (1998): Mean-variance hedging for continuous processes: new proofs and examples. *Finance Stoch.* 2, 173–198.

Protter, P. (2004): *Stochastic integration and differential equations*. Stochastic modelling and applied probability, Springer.

Raible, S. (2000): *Lévy Processes in Finance: Theory, Numerics, and Empirical Facts*. Ph.D. thesis, Freiburg.

Rebonato, R. (2004): *Volatility and Correlation: The Perfect Hedger and the Fox*. Wiley.

Rheinländer, T. (1999): *Optimal martingale measures and their applications in mathematical finance*. Ph.D. thesis, Technischen Universität Berlin.

Rheinländer, T. (2005): An entropy approach to the Stein and Stein model with correlation. *Finance Stoch.* 9, 399–413.

Rheinländer, T. and M. Schweizer (1997): On L^2-projections on a space of stochastic integrals. *The Annals of Probability* 25, 1810–1831.

Rheinländer, T. and G. Steiger (2006): The minimal entropy martingale measure for general Barndorff-Nielsen/Shephard models. *The Annals of Applied Probability* 16, 1319–1351.

Sabanis, S. (2008): Necessary and sufficient conditions for the existence of the q-optimal measure. arXiv:0808.3751v1.

Sato, K. (1999): *Lévy processes and infinitelly divisible distributions.* Cambridge University Press.

Schachermayer, W. (2003): A super-martingale property of the optimal portfolio process. *Finance and Stochastics* 7, 433–456.

Schoutens, W. (2003): *Lévy Processes in Finance: Pricing Financial Derivatives.* John Wiley & Sons, Ltd.

Schweizer, M. (1991): Option hedging for semimartingales. *Stochastic Processes and their Applications* 37, 339–363.

Schweizer, M. (1992a): Martingale densities for general asset prices. *Journal of Mathematical Economics* 21, 363–378.

Schweizer, M. (1992b): Mean-variance hedging for general claims. *The Annals of Applied Probability* 2, 171–179.

Schweizer, M. (1994): Approximating random variables by stochastic integrals. *The Annals of Probability* 22, 1536–1575.

Schweizer, M. (1995): On the minimal martingale measure and the Follmer-Schweizer decomposition. *Stochastic Analysis and Applications* 13, 573–599.

Schweizer, M. (1996): Approximation pricing and the variance optimal martingale measure. *The Annals of Probability* 24, 206–236.

Schweizer, M. (2001): A guided tour through quadratic hedging approaches. In *Option pricing, interest rates and risk management.* Handb. Math. Finance, Cambridge Univ. Press, Cambridge, 538–574.

Steiger, G. (2005): *The optimal martingale measure for investors with exponential utility function.* Ph.D. thesis, ETH Zürich.

Tankov, P. (2004): *Lévy processes in Finance.* Ph.D. thesis, Centre de Mathematiques Appliquees, Ecole Polytechnique, France.

Thierbach, F. (2003): Mean-variance hedging under additional market information. *Int. J. Theor. Appl. Finance* 6, 613–636.

Thomson, R. J. (2005): The pricing of liabilities in an incomplete market using dynamic mean-variance hedging. *Insurance Math. Econom.* 36, 441–455.

Vecer, J. (2001): A New PDE Approach for Pricing Arithmetic Average Asian Options. *The Journal of Computational Finance* 105–113.

Vierthauer, R. (2006): Variance-Optimal Hedging in Affine Stochastic Volatility models. Workshop on Financial Modeling with Jump Processes, Ecole Polytechnique, Palaiseau.

Wiese, A. (1998): *Hedging stochastischer Verpflichtungen in zeitstetigen Modellen.* Verlag Versicherungswirtschaft, Karlsruhe.

Wiesenberg, H. (1998): Modeling market risk in a jump-diffusion setting a generalized Hofmann-Platen-Schweizer-model. Tech. rep., Rheinishe Friedrich-Wilhelms-Universität Bonn.

Wu, X. (2005): *Stochastic volatility with Lévy processes: calibration and pricing.* Ph.D. thesis, University of Maryland.

Xia, J. and J.-A. Yan (2006): Markowitz's portfolio optimization in an incomplete market. *Mathematical Finance* 16, 203–216.

Yanenko, N. (1971): *The Method of Fractional Steps.* Springer-Verlag, Berlin.

Yoeurp, C. and M. Yor (1977): Espace orthogonal a une semi-martingale et applications. Universite Paris VI.